Modal graph theory as a foundation of mathematics

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Modal model theory
- Background

The expressive power of modal graph theory
- Finiteness is expressible
- Countability is expressible
- Size at most continuum $\mathfrak{c}$ is expressible
- Remarks

The interpretative power of modal graph theory
- $\langle H_{\omega_1}, \in\rangle$ is interpretable in modal graph theory
- Stably representable cardinals
- Interpreting V_θ for quite large θ's

Can we interpret V?
- Set-theoretic and meta-mathematical issues

Modal model theory

- Modal model theory injects modal logic into the traditional model theory. We consider a theory T in some first-order language $\mathcal{L}$ and endow it with the substructure relation $\sqsubseteq$. We close $\mathcal{L}$ by Boolean combinations, quantification, and the following modal operators.

Possibility: φ is possible at model M

$M \models \Diamond\varphi$ if there is an $N \sqsupseteq M$ with $N \models \varphi$.

Necessity: φ is necessary at model M

$M \models \Box\varphi$ if all such N have $N \models \varphi$.

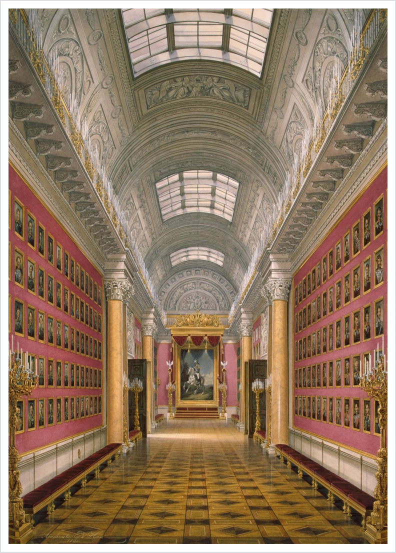
- A graph is a set of vertices and an irreflexive, symmetric binary edge relation $\sim$. Here, $G \sqsubseteq H$ just in case G is an induced subgraph of H.

Example

Let φ be the first-order sentence asserting that there is a 3-cycle. Then, at any graph G,

$$G \models \Diamond\Box\varphi.$$

The expressive power of modal graph theory



Theorem

In the class of graphs, finiteness is expressible in the modal language of graph theory.

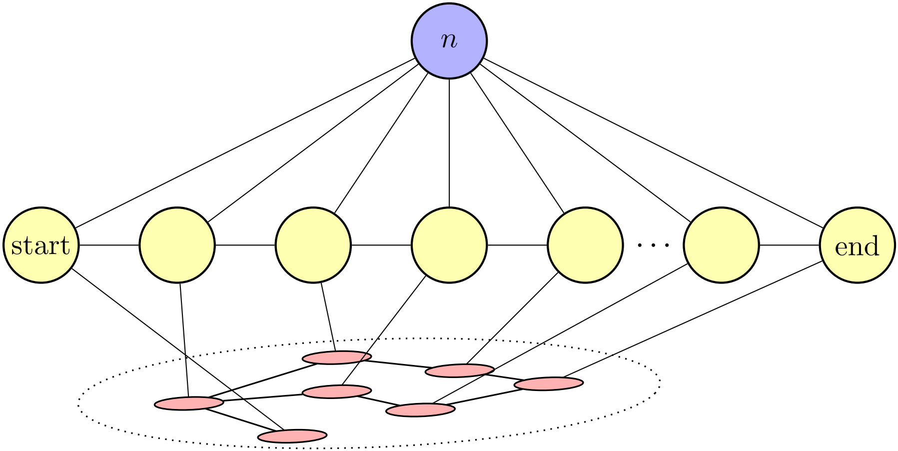
Proof.

An assertion we are looking for is that expresses that *possibly* there is a point n such that:

- neighbor graph of n is connected (this is expressible in modal graph theory);
- it has all vertices of degree 2 within that neighbor set, except exactly two vertices of degree 1 in that neighbor set—a starting node and an ending node;
- all other nodes of the graph are adjacent to exactly one neighbor of n in a bijective correspondence.

The neighbors of n will form a finite chain from the starting vertex to the ending vertex, and the bijection will show that the graph is finite. $\square$

Finiteness is expressible



Theorem

In the class of graphs, countability is expressible in the modal language of graph theory.

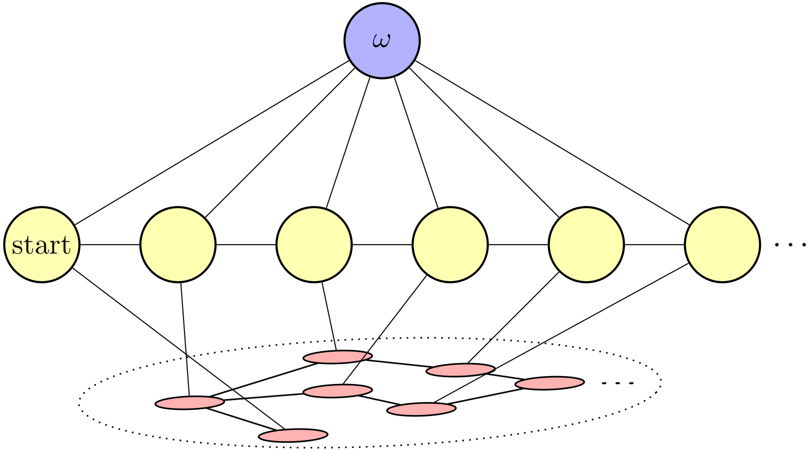
Proof.

We can express that the graph is at most countable by saying that possibly there is a point ω such that:

- neighbor graph of ω is connected;
- it has all vertices of degree 2 except exactly one starting node with degree 1 (amongst the neighbors of ω)—so that these neighbor vertices form an infinite linked chain from the starting node;
- all other nodes in the graph are adjacent to distinct neighbors of ω.

And since finiteness is expressible, one can say that the graph is countable—“it is countable but not finite”. □

Countability is expressible



We shall refer both to the blue node and to the upper half of this graph (subgraph refined to the blue and yellow nodes) as ω .

Theorem

In the class of graphs, the property of having size at most continuum $\mathfrak{c}$ is expressible in the modal language of graph theory.

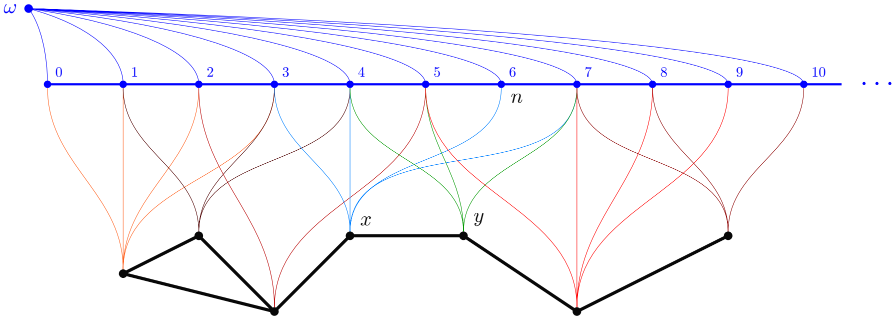
Proof.

We express this by "it is possible that:

- there is a copy of ω;
- for any two distinct nodes x and y not in the copy of ω, there is a neighbor n of ω such that x is adjacent to n if and only if y is not adjacent to n (all other nodes are adjacent to distinct subsets of the neighbors of ω)."

$\square$

Size at most continuum $\mathfrak{c}$ is expressible



Remarks

- The expressive power of modal graph theory is quite remarkable.
- It has the power to express concepts that are fundamental to the study of infinity—finiteness, countability, size at most continuum, and more are coming.
- Needless to say, first-order graph theory falls short of this strength due to compactness and Löwenheim-Skolem theorem.
- On the other hand, modal graph theory is a natural extension of its first-order refinement, preserving its first-order character.

The interpretative power of modal graph theory

Theorem

Hereditarily countable set theory is interpretable in countable modal graph theory. We shall represent hereditarily countable sets with countable graphs and vertices and define a translation $\varphi \mapsto \varphi^*$ of set-theoretic assertions φ to modal graph-theoretic assertions φ^*, such that

$$\langle H_{\omega_1}, \in\rangle \models \varphi(a_0, \ldots, a_n) \quad \Longleftrightarrow \quad \Gamma_0 \oplus \cdots \oplus \Gamma_n \models \varphi^*(\hat{a}_0, \ldots, \hat{a}_n),$$

where $\Gamma \oplus \Lambda$ is the disjoint sum of graphs Γ and Λ and where $(\Gamma_i, \hat{a}_i)$ is a countable graph and vertex representing the set a_i.

Proof.

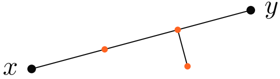
- Every hereditarily countable set is an element of a countable transitive set t;
- the structure $\langle t, \in\rangle$ is a countable set with a well-founded extensional relation $\in$;
- every such well-founded extensional relation on a countable set is isomorphic to a countable transitive set via the Mostowski collapse.

$\langle H_{\omega_1}, \in \rangle$ is interpretable in modal graph theory

Graph code

A *graph code* for a hereditarily countable set is a graph such that:

- it consists of a node t, whose neighbors are related by a well-founded and extensional relation $x \prec y$, defined to hold when there are nodes between x and y as depicted here:



- there is a copy of ω together with a bijection of the copy of ω with the neighbors of t and the nodes used in the $\prec$ relation;
- there is a vertex $\hat{x}$ pointing at t and at one of the neighbors x of t.

$\langle H_{\omega_1}, \in \rangle$ is interpretable in modal graph theory

Being a graph code is expressible in modal graph theory:

- extensionality of $\prec$ is expressible by "distinct neighbors of t have distinct sets of $\prec$-predecessors amongst the neighbors of t;"
- well-foundedness of $\prec$ is expressible by "necessarily, in any extension in which the copy of ω is still a copy of ω and still provides a bijection with the neighbors of t and the supplemental nodes used in the $\prec$ relation, there is no node whose neighbors are a set of neighbors of t having no $\prec$-minimal element."

The graph, together with the vertex $\hat{x}$ code the set that would result from the image of x under the Mostowski collapse of the $\prec$ relation on the neighbors of t.

Interpreting formulas

$x = y$ We can express that two graph codes (Γ, x) and (Λ, y) code the same set by asserting that possibly, there is a node pointing at nodes that are each adjacent to a node in Γ and to a node in Λ in such a way that they make a isomorphic correspondence between $\prec$-downward closed subsets of Γ and Λ, which furthermore associates x with y. The set coded by x via Γ will be the same as the set coded by y via Λ;

$x \in y$ similarly, we can express that one code (Γ, x) codes an element of another (Λ, y), if (Γ, x) is isomorphic to the code formed by a $\prec$-predecessor of y in Λ;

We extend to arbitrary formulas recursively.

Definition

We say that a cardinal κ is *stably representable* if there is a property ϕ expressible in modal graph theory with the following properties:

1. There is a graph G with a vertex v satisfying $\phi(v)$.
2. If $\phi(v)$ holds in a graph G of a vertex v, then v has exactly κ many neighbors in G.
3. The truth of $\phi(v)$ in G depends only on the induced subgraph consisting of v and its neighbors and the neighbors of the parameters (suppressed).
4. If $\phi(v)$ holds in G and also in an extension graph H, then v has the same neighbors in G as in H.

Observation

ω is stably representable.

Theorem

If a cardinal κ is stably representable, then $\langle H_{\kappa^+}, \in\rangle$ is interpretable in modal graph theory.

Proof.

- In place of a bijection with a copy of ω, we use a stable representation of κ;
- we represent sets in H_{κ^+} using well-founded extensional relations on sets of size κ, with a distinguished vertex pointing at the set being represented;
- we can express the equivalence of codes and the set membership relation on the codes in the language of modal graph theory.

$\square$

Theorem

If κ is stably representable, then so is κ^+, 2^κ, $\aleph_\kappa$, $\beth_\kappa$, next $\beth$-fixed-point above κ, the first $\beth$-hyper-fixed-point above κ, the first $\beth$-hyper-hyper-fixed-point above κ, and so on.

Corollary

Set-theoretic truth in V_θ, where θ is the first $\beth$-hyper-fixed point, is interpretable in modal graph theory.

- We may proceed similarly to the next $\beth$-hyper-fixed point, the next hyper-hyper-fixed point, and so on. These will all also be stably representable.
- And so we can interpret set-theoretic truth for a quite a long way into the cumulative hierarchy.
- In this sense, modal graph theory can serve as a foundation of mathematics.
- But does it ever stop?

Question

Is set-theoretic truth, that is, truth in the full set-theoretic universe V, interpretable in modal graph theory?

- The answer is *positive* if we extend the language by means of the actuality operator @, which allows one to refer to the actual world and more generally to the various worlds that are in effect referenced during the course of interpreting a modal statement.

Key obstacle

Can one find sufficient stable representations of any given set.

- It is not clear how to translate definable cardinals in ZFC, such as the first Σ_3-correct cardinal, into stable representations in modal graph theory.

Can we express that a relation is well-ordered?

- The ability to express the well-order concept in modal graph theory seems likely to be highly relevant for the capacity of modal graph to interpret set-theoretic truth.

Equinumerosity problem
: Can we express in the language of modal graph theory that the neighbor set G_x of vertex x is equinumerous with the neighbor set G_y of vertex y?

Cardinal comparability problem
: Can we express in the language of modal graph theory that the neighbor set G_x of vertex x has cardinality less than or equal to the neighbor set G_y of vertex y?

- $\mathrm{Mod}(T)$ is a proper class and the recursive definition of truth is not a set-like recursion;
- Therefore, we can't seem to undertake the definition legitimately in ZFC;

Question

In ZFC *can one define the satisfaction relation for modal graph theory for the class of all graphs?*

- This question is related to the question whether modal graph theory interprets set-theoretic truth.
- If it does, then the answer to this question will be negative, since by Tarski's theorem on the non-definability of truth one cannot define first-order set-theoretic truth within first-order set theory.

[HW20] Joel David Hamkins and Wojciech Aleksander Wołoszyn. "Modal model theory". In: *Mathematics arXiv* (2020). Under review. arXiv: 2009.09394 [math.LO].