Modal model theory

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Modal model theory

We take a structure M in some first order language $\mathcal{L}$ and consider it within a class $\mathcal{C}$ of similar structures, endowing the class with an accessibility relation $\Rightarrow$.
We can now view $(\mathcal{C}, \Rightarrow)$ as a Kripke model of possible worlds, giving rise to the natural modal operators:

Possibility: φ is possible at structure $M \in \mathcal{C}$

$M \models \Diamond\varphi$ if there is an $N \in \mathcal{C}$ with $M \Rightarrow N$ and $N \models \varphi$.

Necessity: φ is necessary at structure $M \in \mathcal{C}$

$M \models \Box\varphi$ if all such N have $N \models \varphi$.

- Modal model theory turns out to be a quite prolific area of expertise.
- It acts as an interesting extension of the classical model theory. There is a version of Löwenheim-Skolem, Łoś theorem, and more.
- There is a plethora of possibilities for case studies: sets, groups, fields, graphs, linear orderds, or what have you, all viewed under homomorphisms, embeddings, "reverse embeddings," etc.

Graphs under embeddings

Let $(\mathrm{Mod}(T), \subseteq)$ be the class of all graphs under embeddings and φ a first order sentence asserting that there is a 3-cycle. Then any graph G satisfies $\Diamond\Box\varphi$.

Categories of $\mathcal{L}$-structures

Any category of $\mathcal{L}$-structures $\mathcal{C}$ can be understood in a Kripkean manner—morphisms can act as an accessibility relation. If $\mathcal{C}$ is pointed (there is a zero-object), then all individuals are possibly necessarily identical. At any object M, we have that $M \models \forall x \forall y \Diamond\Box(x = y)$.

Groups under "reversed embeddings"

Here, a group G accesses another group H just in case H is a subgroup of G—a fun example where individuals cease to exist, and one has to decide how to asserts things about non-existent individuals, i.e. does $x = y$ if x and y do not exist?

Ongoing research

A paper investigating modal model theory under substructure/embeddability relation, building on the case of graphs, is under review:

[HW20] Joel David Hamkins and Wojciech Aleksander Wołoszyn. "Modal model theory". In: *Mathematics arXiv* (2020). Under review. arXiv: 2009.09394 [math.LO].

There is a significant progress on:

- the case study of modal theories of groups, fields, and linear orders, soon to be submitted,
- general results expanding [HW20],

and a moderate progress regarding:

- philosophical discussion on a correct semantics for modal model theory (what to do about objects that cease to exist), and
- modal theory of categories of $\mathcal{L}$-structures.