The Mathematical Multiverse

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Outline

No man is an Iland, intire of itselfe; every man
is a peece of the Continent, a part of the maine;
if a Clod bee washed away by the Sea, Europe
is the lesse, as well as if a Promontorie were, as
well as if a Manor of thy friends or of thine
owne were; any mans death diminishes me,
because I am involved in Mankinde;
And therefore never send to know for whom
the bell tolls; It tolls for thee.

John Donne

MEDITATION XVII

Devotions upon Emergent Occasions

No structure is an island—we want to speak of mathematical structures in the context of other mathematical structures.

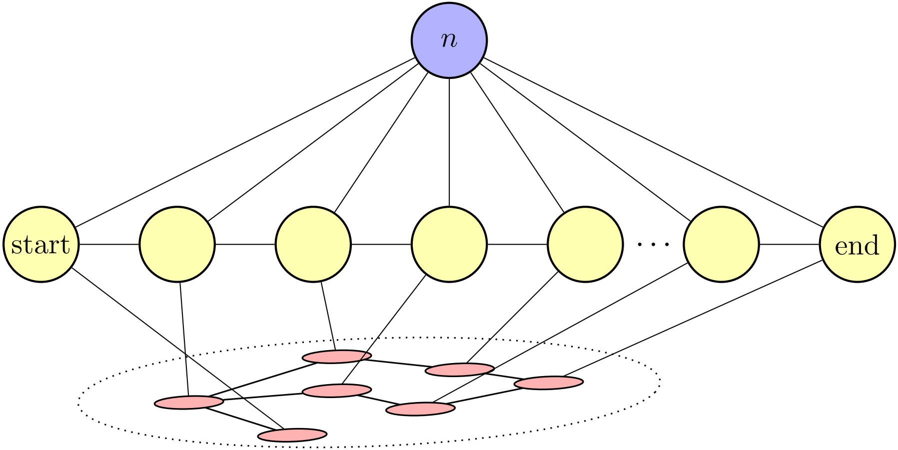
- Consider the collection $\mathcal{V}$ of all sets together with all functions between them.
- Arguably, $\mathcal{V}$ carries all the data about mathematical structures, the *worlds* of $\mathcal{V}$, and how these worlds relate to each other—one can recover any relation R on a set $W \in \mathcal{V}$ and see what $f(R) \subset V^n$ is for some $f\colon W \to V$, where $V \in \mathcal{V}$.
- One can reduce ontological commitments of $\mathcal{V}$ by asking about modal commitments of the worlds within it.

Example

Collection $\mathcal{V}$ knows everything about the class $\mathcal{G}$ of all graphs $(G, \sim)$ together with all embeddings between them.

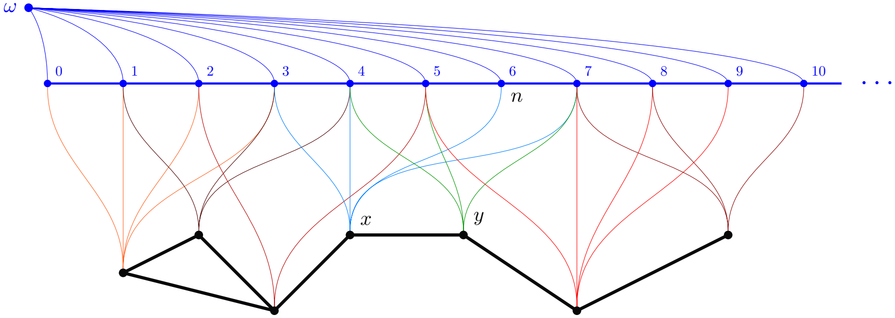
- Define the semantics recursively in the usual Tarskian manner but with an additional notion of possibility. More precisely, $G \models_{\mathcal{G}} \Diamond \varphi$ just in case there is a graph embedding $e \colon G \to H$ such that $H \models_{\mathcal{G}} \varphi$.
- Suppose it is possible for a graph to have a point n whose neighborhood graph is connected and has all vertices of degree 2 except exactly two nodes of degree 1, *start* and *end*. Moreover, all other nodes of the graph are adjacent to exactly one neighbor of n in a bijective correspondence.

No graph is an island



- But this is a first-order assertion expressing that a graph is finite! To answer if this assertion possibly fails is to answer whether infinity exists or not.
- Can we push it further...
- to continuum $\mathfrak{c}$ or first uncountable ordinal ω_1?

Size at most continuum $\mathfrak{c}$ is expressible



Theorem

Set-theoretic truth in V_θ, where θ is the first $\beth$-fixed point, is interpretable.

- Holds similarly for the next $\beth$-fixed point, the first hyper-hyper-fixed point, and so on.
- We can interpret set-theoretic truth for a quite a long way into the cumulative hierarchy.
- Perhaps one can even interpret truth in V.
- And graphs under embeddings are only a prototypical example for a modal model-theorist to study.

Modal model theory

Definition

A *Kripke category* is a class of $\mathcal{L}$-structures, called *worlds*, that are interconnected by functions, called *accessibility maps*, such that

- the identity function on any world within that class is an accessibility map for that class, and
- the composition of an accessibility map from world W to world V and an accessibility map from world V to world U, is an accessibility map from world W to world U.

Example

Class of all groups under homomorphisms can be seen as a Kripke category.

Diverse Kripke categories

- Under homomorphisms, strong homomorphisms, embeddings, substructures:
  - graphs,
  - groups,
  - fields,
  - any type of orderings.
- Under end-extensions:
  - linear orders,
  - arithmetic,
  - set theory.
- Diverse natural transformations on $\mathrm{Mod}(T)$ for any theory T.

Goals of this project

- To investigate the modal commitments of the mathematical multiverse $\mathcal{V}$ or fragments thereof upon endowing its worlds with a common notion of a structure.
- To investigate various specific natural Kripke categories.
- To find new modal axioms and principles, expressed in the modal language.

Let $\mathcal{K}$ be a Kripke category and s denote a function that assigns variables to individuals in a world.

Possibility: φ is possible at world W in $\mathcal{K}$

$W \models_{\mathcal{K}} \Diamond \varphi[s]$ if there is an accessibility mapping $f\colon M \to N$ to some world N with $N \models_{\mathcal{K}} \varphi[f \circ s]$.

Necessity: φ is necessary at world W in $\mathcal{K}$

$W \models_{\mathcal{K}} \Box \varphi[s]$ if all accessibility mapping $f\colon M \to N$ to each such N's have $N \models_{\mathcal{K}} \varphi[f \circ s]$.

Example

In a Kripke category $\mathcal{K}$ of all groups under homomorphisms,

$$G \models_{\mathcal{K}} \forall x \, \Diamond (x = e)$$

for any group G with a neutral element e.

Modal validities

It is easy to see that S4 is valid at every world in any Kripke category.

$$
\begin{array}{rl}
\mathrm{K} & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
\mathrm{S} & \Box\varphi \to \varphi \\
4 & \Box\varphi \to \Box\Box\varphi \\
.2 & \Diamond\Box\varphi \to \Box\Diamond\varphi \\
.3 & (\Diamond\varphi \wedge \Diamond\psi) \to \Diamond\big((\varphi \wedge \Diamond\psi) \vee (\Diamond\varphi \wedge \psi)\big) \\
5 & \Diamond\Box\varphi \to \varphi.
\end{array}
$$

S4 is obtained by closing under modus ponens and necessitation.

$$
\begin{tikzcd}
 & N & \\
N_0 \arrow[ur, dashed] & & N_1 \arrow[ul, dashed] \\
 & M \arrow[ul, "f_0"'] \arrow[ur, "f_1"] &
\end{tikzcd}
$$

convergent

$$
\begin{tikzcd}
N_0 \arrow[rr, dashed] & & N_1 \\
 & M \arrow[ul, "f_0"'] \arrow[ur, "f_1"] &
\end{tikzcd}
$$

connectable

$$
\begin{tikzcd}
 & N & \\
N_0 \arrow[ur, dashed] & \circlearrowleft & N_1 \arrow[ul, dashed] \\
 & M \arrow[ul, "f_0"'] \arrow[ur, "f_1"] &
\end{tikzcd}
$$

amalgamable

$$
\begin{tikzcd}
N_0 \arrow[rr, dashed] & & N_1 \\
 & M \arrow[ul, "f_0"'] \arrow[ur, "f_1"] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}
$$

factorable

For Kripke categories $\mathrm{Emb}(T)$ and $\mathrm{Hom}(T)$—models of theory T under embeddings and homomorphisms, respectively—the following characterization holds:

Theorem

- S4.2 *is valid for sentences, with no parameters allowed, just in case each span is convergent.*
- S4.2 *is valid for formulas, even with parameters, just in case each span is amalgamable.*

Remark

No analogue of this characterization for S4.3 *and connectability or factorability. The backwards direction, however, works in any Kripke category both for* S4.2 *and* S4.3.

M $\quad \Box\Diamond\varphi \to \Diamond\Box\varphi.$

Valid for graph theory under inclusions?

No. It is necessarily possible that the diameter is two. You can extend any graph to make it true. But this can also be destroyed by adding isolated points.

H $\quad \Diamond\Box(\varphi \leftrightarrow \Box\varphi).$

Valid for **Inj**?

Yes. The condition $\varphi \leftrightarrow \Box\varphi$ is called *modality trivialization*. Every formula about **Inj** is equivalent to a Boolean combination of the equality and non-equality patterns of finitely many variables and first-order sentences expressing that there are at least n elements.

More advanced results

More advanced results

Further results that I should like to mention concern:

- Finding lower and upper bounds on propositional modal validities of natural Kripke categories such as **Set**, **Inj**, **Groups**, **Fields**, linearly ordered sets under diverse accessibility mappings, and so on. For instance, both **Groups** and **Fields** under embeddings have attainable lower and upper bounds S4.2 and S5, respectively.
- *Modality elimination* and its relationship with quantifier elimination, model completeness, and model companions. One can, for example, give precise modal axioms that are equivalent to being a model companion.
- Characterizing worlds validating the maximality principle S5, in interesting Kripke categories. A prototypical example here would be algebraically closed fields in **Fields** under embeddings.
- Expressive and interpretative powers of specific Kripke categories. In groups under homomorphisms, one can interpret arithmetic on the integers.

Each of these research efforts requires a robust technology of its own, e.g., the *control statements* technique for finding precise bounds on propositional modal validities. Perhaps I shall explain them more fully next term or today in the pub. Some of these ideas have already surfaced in my paper with Joel David Hamkins.

[HW20] Joel David Hamkins and Wojciech Aleksander Wołoszyn. "Modal model theory". In: *Mathematics arXiv* (2020). Under review. arXiv: `2009.09394 [math.LO]`.

Thank you!

Slides and further readings available at www.woloszyn.org.