Kripke category of sets

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June 7, 2022

Outline

From concrete to Kripke categories
- What is a Kripke category?
- Kripke semantics
- Validity in Kripke categories
- Well-established modal principles
- Common modal theories

Kripke category of sets
- The archetypical Kripke category
- Salient subcategories of Set
- Main results

Grzegorczyk axiom
- Deciphering Grz
- The case of buttons but no switches
- Grz.2 as a modal theory of infinite worlds in $\mathrm{Set}_{\twoheadrightarrow}$

Outro

What is a Kripke category?

A *Kripke category* is a concrete category of structures in a common first-order language $\mathcal{L}$. Each object in the category is an $\mathcal{L}$-structure and each morphism is an $\mathcal{L}$-homomorphisms. We call these objects *worlds* and morphisms *accessibility mappings*.

- Any concrete category is a Kripke category in the language of equality $\mathcal{L}_=$, comprised only of logical symbols and a symbol for equality.
- Many concrete categories are Kripke categories in their more expressive inherent first-order languages. For instance, the category of groups Groups is a Kripke category in the language of first-order group theory $\mathcal{L}_{\langle \odot, e \rangle}$.

Kripke semantics

Any such Kripke category, I claim, gives rise to natural modal semantics.

- $W \models_{\mathcal{K}} \Diamond \varphi[\nu]$ if there is an accessibility mapping $f\colon W \to U$ in $\mathcal{K}$ such that $U \models_{\mathcal{K}} \varphi[f \circ \nu]$.
- $W \models_{\mathcal{K}} \Box \varphi[\nu]$ if for any accessibility mappings $f\colon W \to U$ in $\mathcal{K}$, we have that $U \models_{\mathcal{K}} \varphi[f \circ \nu]$.

The project is to investigate the resulting modal theories and validities in a variety of cases.

A propositional modal assertion $\varphi(p_0, \ldots, p_n)$, is *valid* at a world W in a Kripke category $\mathcal{K}$ with respect to a language of substitution instances, if all substitution instances $\varphi(\psi_0, \ldots, \psi_n)$ arising for assertions ψ_i in that language of substitutions are true at W in $\mathcal{K}$.

Observation

The modal principle 4, $\Box p \to \Box\Box p$, *is valid at any world in any Kripke category. This is because morphisms are closed under compositions in any category.*

Well-established modal principles

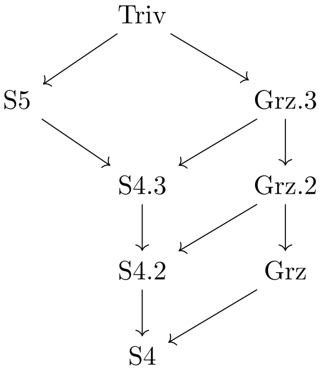
The following is a non-exhaustive list of well-established propositional modal principles.

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
Dual	$\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$
B	$\varphi \to \Box\Diamond\varphi$
4	$\Box\varphi \to \Box\Box\varphi$
T	$\Box\varphi \to \varphi$
Grz	$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$
.2	$\Diamond\Box\varphi \to \Box\Diamond\varphi$
.3	$\Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \to \varphi)$
5	$\Diamond\Box\varphi \to \varphi$
Triv	$\Box\varphi \leftrightarrow \varphi$

Common modal theories

Let us list important modal theories that arise from the aforementioned well-established modal principles. Here, "+" necessitates that the resulting theory is *normal*, and particularly closed under modus ponens and necessitation.

$$
\begin{aligned}
\mathrm{Triv} &= \mathrm{Triv} + 4 = \mathrm{S5} + \mathrm{Grz} \\
\mathrm{S5} &= \mathrm{S4} + 5 = \mathrm{S4} + \mathrm{B} \\
\mathrm{S4.3} &= \mathrm{S4} + .3 \\
\mathrm{S4.2} &= \mathrm{S4} + .2 \\
\mathrm{Grz.3} &= \mathrm{S4.3} + \mathrm{Grz} \\
\mathrm{Grz.2} &= \mathrm{S4.2} + \mathrm{Grz} \\
\mathrm{Grz} &= \mathrm{K} + \mathrm{Grz} = \mathrm{S4} + \mathrm{Grz} \\
\mathrm{S4} &= \mathrm{K} + 4
\end{aligned}
$$



The archetypical Kripke category

Just like the category of sets is the archetypical concrete category, the category of sets in the language of equality $\mathcal{L}_=$ is the archetypical Kripke category. All Kripke categories are built upon Set, by either:

- simultaneous model expansion of all worlds,
- world annihilation, or
- accessibility removal.

Salient subcategories of Set

We distinguish the following subcategories of Set, arising from natural types of morphisms:

- $\rightarrow$ Kripke category of sets and functions Set,
- $\rightarrowtail$ Kripke category of sets and injective functions $\mathrm{Set}_{\rightarrowtail}$,
- $\hookrightarrow$ Kripke category of sets and inclusions $\mathrm{Set}_{\hookrightarrow}$,
- $\twoheadrightarrow$ Kripke category of sets and surjective functions $\mathrm{Set}_{\twoheadrightarrow}$,
- $\rightarrowtail\!\!\!\!\twoheadrightarrow$ Kripke category of sets and bijective functions $\mathrm{Set}_{\rightarrowtail\!\!\twoheadrightarrow}$,
- $\hookrightarrow\!\!\!\!\twoheadrightarrow$ Kripke category of sets and identities $\mathrm{Set}_{\hookrightarrow\!\!\twoheadrightarrow}$.

Main results

	Propositional modal validities			
	Finite worlds		Infinite worlds	
Kripke category Set	Sentential	Formulaic	Sentential	Formulaic
$\to$ Homomorphisms	S5	S5/ Fin	S5	S4.2
$\twoheadrightarrow$ Epimorphisms	Triv/ Fin	Triv/ Fin	Grz.3	Grz.2
$\rightarrowtail$ Monomorphisms	Grz.3	Grz.3	Triv	Triv
$\hookrightarrow$ Inclusions	Grz.3	Grz.3	Triv	Triv
$\rightarrowtail\!\!\!\!\twoheadrightarrow$ Isomorphisms	Triv	Triv	Triv	Triv
$\hookrightarrow\!\!\!\!\twoheadrightarrow$ Identities	Triv	Triv	Triv	Triv

What I mean by $T/$ Fin is that T is valid save for finitely many substitution instances. This manner of exposition is only to save time and space. We can give precise modal validities in each case, oftentimes depending on the size of the underlying world.

Let us try to decipher what the Grzegorczyk axiom says...

$$\text{Grz} \quad \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$$

Can you do it?

We say that an assertion φ is a *penultimate* truth at W just in case $W \models \neg\varphi \wedge \Box(\varphi \to \Box\varphi)$.

Theorem

Suppose that W_0 is a world in a Kripke category $\mathcal{K}$ in a modal language $\mathcal{L}^{\Diamond}$. The propositional modal theory Grz *is valid at W_0 with respect to a language of substitution instances if and only if each assertion in that language of substitutions is either true or possibly penultimate on the cone above W_0.*

The case of buttons but no switches

Definition

- A *button* is a statement that is necessarily possibly necessary. It is *unpushed* if it is not true yet.
- A *switch* is a statement that one can always turn on and off, as one pleases.
- Control statements on a given list are *independent* if fiddling with statements from any sublist does not interfere with any other statement on the list.

In May 2019, Joel David Hamkins and I had claimed that arbitrarily many independent unpushed buttons suffice to bound the validities of a potentialist system (Kripke category) to the modal logic S4.2, what we later retracted. Meanwhile, Robert Passmann asked about the "proof," and what can be salvaged in case it is wrong. In my work, I show that the optimal upper bound for buttons is the modal theory Grz.2 (together with similar results for Grz.3 and Grz), answering Robert's MathOverflow question.

The case of buttons but no switches

Theorem

Suppose W is a world in a Kripke category $\mathcal{K}$ that has arbitrarily large finite families of independent buttons. Then, the propositional modal validities are contained within the propositional modal theory Grz.2*.*

Proof sketch.

- It is known that Grz.2 is characterized by finite partial orders.
- Show that Grz.2 is characterized by finite lattices/Boolean algebras.
  - If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model M_0 wherein φ fails.
  - Use partial tree unraveling to construct a baled tree M bisimilar to M_0.
- provide an assignment of the propositional variables p_i to assertions ψ_{p_i} in the modal language, so that $(M, w) \models \varphi(p_0, \ldots, p_n)$ just in case $W \models \varphi(\psi_{p_0}, \ldots, \psi_{p_n})$.
- φ fails at w, so $W \models \neg\varphi(\psi_{p_0}, \ldots, \psi_{p_n})$, and Grz.2 cannot be valid at W.

Theorem

The propositional modal validities of infinite worlds in the Kripke category of sets and surjective functions $\mathrm{Set}_{\twoheadrightarrow}$, *constitute precisely the propositional modal theory* Grz.2*.*

Hint.

All infinite sets have the same modal truths. □

Lower bounds proof sketch.

- Let W be any world and $\varphi[\bar{a}]$ an assertion with $\bar{a} \in W^n$.
- If $W \models \varphi[\bar{a}]$, then we are done. If not, then either $W \models \Box\neg\varphi[\bar{a}]$, in which case we are done as well, or there is a non-empty family $\mathcal{U}$ of worlds and accessibility mappings $W \xrightarrow{f} U$ such that $U \models \varphi[f(\bar{a})]$.
- But there must be a $U \in \mathcal{U}$ of the smallest finite size, because all infinite sets have the same modal truths. That world witnesses the possible penultimacy of $\varphi[\bar{a}]$ at W.

□

Stay tuned for upcoming articles with more details and additional results. There will be Kripke categories of finite sets, linear orders, groups, or what have you.

Thank you!

Slides and further readings available at www.woloszyn.org.

Other recent work on modal logic and modal model theory

[Ada22] Sam Adam-Day. "Bisimulations of potentialist systems". In: (2022). Under review.

[Ham18] Joel David Hamkins. "The modal logic of arithmetic potentialism and the universal algorithm". In: *ArXiv e-prints* (2018). Under review, pp. 1–35. arXiv: 1801.04599 [math.LO]. URL: http://wp.me/p5M0LV-1Dh.

[HL19] Joel David Hamkins and Øystein Linnebo. "The modal logic of set-theoretic potentialism and the potentialist maximality principles". In: *Review of Symbolic Logic* (October 2019). DOI: 10.1017/S1755020318000242. arXiv: 1708.01644 [math.LO]. URL: http://wp.me/p5M0LV-1zC.

[HW20] Joel David Hamkins and Wojciech Aleksander Wołoszyn. "Modal model theory". In: *Mathematics arXiv* (2020). Under review. arXiv: 2009.09394 [math.LO].

[SS20] Denis I. Saveliev and Ilya Shapirovsky. "On Modal Logics of Model-Theoretic Relations". In: *Stud Logica* 108 (2020), pp. 989–1017.