Usuba's extendible and resurrection

supervised by W. Hugh Woodin

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Resurrection axiom

Let $\Diamond\varphi$ mean that φ is *forceable*, that is true in some forcing extension. And let $\Box$ be an operator dual to $\Diamond$, which is to say $\Box\varphi$ holds just in case $\neg\varphi$ is not forceable.

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For a class of assertions Γ, the Γ-*resurrection axiom* with real parameters $\mathrm{RA}_\Gamma(\mathbb{R})$ is the resurrection axiom with real parameters qualified to formulas $\varphi(x)$ from the class Γ.

Theorem (Woodin)

If there is a proper class of Woodin cardinals, then the Σ_2-resurrection axiom with real parameters $\mathrm{RA}_{\Sigma_2}(\mathbb{R})$ *holds.*

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Observation

Suppose there is one I0 cardinal λ and a proper class of Woodin cardinals. We can collapse λ, so there is only a proper class of Woodin cardinals and no I0 cardinals. But then we can resurrect the existence of I0 in any further generic extension.

Suppose M is a transitive inner model of ZFC. We say that M is a *ground* just in case there is a poset $\mathbb{P} \in M$ and an $(M, \mathbb{P})$-generic $G \subseteq \mathbb{P}$ such that $M[G] = V$. The *mantle* is the intersection of all grounds of V. Usuba proved that the mantle is a model of ZFC.

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Theorem (Usuba)

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Theorem (Goldberg)

It is consistent that Usuba's theorem fails in V_κ where κ is the least extendible cardinal.

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Theorem (Woodin)

If $(\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}} + \Theta$ *is regular*$)$ *is consistent, then so is* $\mathrm{ZFC} + \mathrm{RA}(\mathbb{R})$, *where* Θ *is the least non-zero ordinal such that there is no surjection from the reals onto it.*

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- show that the tension between large cardinals and resurrection is about the complexity strength as opposed to the consistency strength, and
- find an optimal stage in the complexity hierarchy where resurrection becomes inconsistent with the existence of large cardinals.

As a preview, let me briefly discuss the following theorem.

Theorem

Suppose the mantle $\mathbb{M}$ *is a ground. Then the* Π_3*-resurrection axiom with real parameters* $\mathrm{RA}_{\Pi_3}(\mathbb{R})$ *fails.*

Proof sketch.

- For $V_\gamma \prec_{\Sigma_2} V$, we get that $\mathbb{M} \cap V_\gamma$ is the mantle of V_γ.

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- In the generic extension, say $V[H]$, the mantle is still a ground but ω_1 is now the successor of the mantle.

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- Take $\gamma > \lambda$ with $V_\gamma \prec_{\Sigma_2} V$. The mantle of $V[H]_\gamma$ is the real mantle—$\mathbb{M}^{V[H]_\gamma} = \mathbb{M} \cap V[H]_\gamma$.

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- Let φ say that for any sufficiently large α with $V_\alpha \prec_{\Sigma_2} V$, ω_1 is the least cardinal of the mantle that is larger than the real coding λ.

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- Let φ say that for any sufficiently large α with $V_\alpha \prec_{\Sigma_2} V$, ω_1 is the least cardinal of the mantle that is larger than the real coding λ.
- Thus, one can force the failure of φ but cannot resurrect it.

$\square$

Thank you!

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[FHR15] Gunter Fuchs, Joel David Hamkins, and Jonas Reitz. "Set-theoretic geology". In: *Annals of Pure and Applied Logic* 166.4 (2015), pp. 464–501. ISSN: 0168-0072. DOI: `10.1016/j.apal.2014.11.004`. arXiv: `1107.4776 [math.LO]`. URL: `http://jdh.hamkins.org/set-theoreticgeology`.

[Gol21] Gabriel Goldberg. *Usuba's extendible cardinal.* 2021. URL: `https://arxiv.org/abs/2108.06903`.

[Lar04] Paul B Larson. *The stationary tower : notes on a course given by W. Hugh Woodin.* eng. University lecture series (Providence, R.I.) ; 32. Providence, R.I.: American Mathematical Society, 2004. ISBN: 9780821836040.

[Usu19] Toshimichi Usuba. "Extendible cardinals and the mantle". In: *Archive for Mathematical Logic* 58 (February 2019). DOI: `10.1007/s00153-018-0625-4`.

[Woo21] W. Hugh Woodin. Personal communication. June 2021.