# Usuba's extendible and resurrection supervised by W. Hugh Woodin



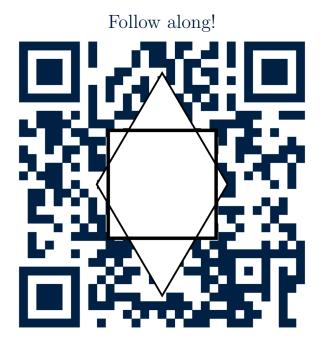


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For a class of assertions  $\Gamma$ , the  $\Gamma$ -resurrection axiom with real parameters  $\operatorname{RA}_{\Gamma}(\mathbb{R})$  is the resurrection axiom with real parameters qualified to formulas  $\varphi(x)$  from the class  $\Gamma$ .





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#### Observation

Suppose there is one I0 cardinal  $\lambda$  and a proper class of Woodin cardinals. We can collapse  $\lambda$ , so there is only a proper class of Woodin cardinals and no I0 cardinals. But then we can resurrect the existence of I0 in any further generic extension.



Suppose M is a transitive inner model of ZFC. We say that M is a ground just in case there is a poset  $\mathbb{P} \in M$  and an  $(M, \mathbb{P})$ -generic  $G \subseteq \mathbb{P}$  such that M[G] = V. The mantle is the intersection of all grounds of V. Usuba proved that the mantle is a model of ZFC.

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Theorem (Usuba)

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#### Theorem (Usuba)

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## Theorem (Goldberg)

It is consistent that Usuba's theorem fails in  $V_{\kappa}$  where  $\kappa$  is the least extendible cardinal.



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# Theorem (Woodin)

If  $(ZF + AD_{\mathbb{R}} + \Theta \text{ is regular})$  is consistent, then so is  $ZFC + RA(\mathbb{R})$ , where  $\Theta$  is the least non-zero ordinal such that there is no surjection from the reals onto it.



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- ► show that the tension between large cardinals and resurrection is about the complexity strength as opposed to the consistency strength, and
- ▶ find an optimal stage in the complexity hierarchy where resurrection becomes inconsistent with the existence of large cardinals.

As a preview, let me briefly discuss the following theorem.

#### Theorem

Suppose the mantle  $\mathbb{M}$  is a ground. Then the  $\Pi_3$ -resurrection axiom with real parameters  $\operatorname{RA}_{\Pi_3}(\mathbb{R})$  fails.



Proof sketch.

• For  $V_{\gamma} \prec_{\Sigma_2} V$ , we get that  $\mathbb{M} \cap V_{\gamma}$  is the mantle of  $V_{\gamma}$ .



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- ► Take  $\gamma > \lambda$  with  $V_{\gamma} \prec_{\Sigma_2} V$ . The mantle of  $V[H]_{\gamma}$  is the real mantle— $\mathbb{M}^{V[H]_{\gamma}} = \mathbb{M} \cap V[H]_{\gamma}$ .

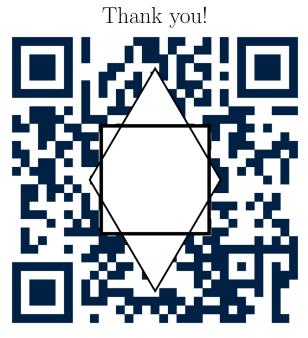


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- Let  $\varphi$  say that for any sufficiently large  $\alpha$  with  $V_{\alpha} \prec_{\Sigma_2} V$ ,  $\omega_1$  is the least cardinal of the mantle that is larger than the real coding  $\lambda$ .

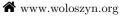


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- Let  $\varphi$  say that for any sufficiently large  $\alpha$  with  $V_{\alpha} \prec_{\Sigma_2} V$ ,  $\omega_1$  is the least cardinal of the mantle that is larger than the real coding  $\lambda$ .
- ▶ Thus, one can force the failure of  $\varphi$  but cannot resurrect it.









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