# Modal model theory

supervised by Joel David Hamkins


Wojciech Aleksander Wołoszyn

@WAWoloszyn www.woloszyn.org

Mathematical Institute
University of Oxford

St Hilda's College
Oxford



IMS Graduate Summer School in Logic
19th July 2022

# Follow along!

This talk includes

- joint work with Joel David Hamkins, and
- work supervised by Joel David Hamkins.

# Introducing modal model theory

In modal model theory, we consider a mathematical structure within the context of a class of similar structures.

In modal model theory, we consider a mathematical structure within the context of a class of similar structures.

A *Kripke category* is a concrete category of structures in a common first-order language $\mathcal{L}$, so as to be considered as a Kripke model of possible worlds and analyzed by the means of tools and techniques in modal logic.

In modal model theory, we consider a mathematical structure within the context of a class of similar structures.

A *Kripke category* is a concrete category of structures in a common first-order language $\mathcal{L}$, so as to be considered as a Kripke model of possible worlds and analyzed by the means of tools and techniques in modal logic.

Each object in the category is an $\mathcal{L}$-structure and each morphism is an $\mathcal{L}$-homomorphism. We call these objects *worlds* and the morphisms *accessibility mappings*.

Examples:

Examples:

- graphs, groups, fields,

Examples:

- graphs, groups, fields,
- orders, linear orders

Examples:

- graphs, groups, fields,
- orders, linear orders
- models of PA,

Examples:

- graphs, groups, fields,
- orders, linear orders
- models of PA,
- models of set theory.

Examples:

- graphs, groups, fields,
- orders, linear orders
- models of PA,
- models of set theory.

A central case for modal model theory is the class $\mathrm{Mod}(T)$ of all models of first-order theory $T$ under natural classes of $\mathcal{L}$-homomorphisms:

Examples:

- graphs, groups, fields,
- orders, linear orders
- models of PA,
- models of set theory.

A central case for modal model theory is the class $\mathrm{Mod}(T)$ of all models of first-order theory $T$ under natural classes of $\mathcal{L}$-homomorphisms:

- all homomorphisms,

Examples:

- graphs, groups, fields,
- orders, linear orders
- models of PA,
- models of set theory.

A central case for modal model theory is the class $\mathrm{Mod}(T)$ of all models of first-order theory $T$ under natural classes of $\mathcal{L}$-homomorphisms:

- all homomorphisms,
- all monomorphisms,

Examples:

- graphs, groups, fields,
- orders, linear orders
- models of PA,
- models of set theory.

A central case for modal model theory is the class $\mathrm{Mod}(T)$ of all models of first-order theory $T$ under natural classes of $\mathcal{L}$-homomorphisms:

- all homomorphisms,
- all monomorphisms,
- all epimorphisms.

We extend the standard Tarskian semantics as follows.

We extend the standard Tarskian semantics as follows.

- $W \models \Diamond\varphi[\nu]$ if there is an accessibility mapping $f\colon W \to U$ such that $U \models \varphi[f \circ \nu]$;

We extend the standard Tarskian semantics as follows.

- $W \models \Diamond\varphi[\nu]$ if there is an accessibility mapping $f\colon W \to U$ such that $U \models \varphi[f \circ \nu]$;
- $W \models \Box\varphi[\nu]$ if for any accessibility mappings $f\colon W \to U$, we have that $U \models \varphi[f \circ \nu]$.

We extend the standard Tarskian semantics as follows.

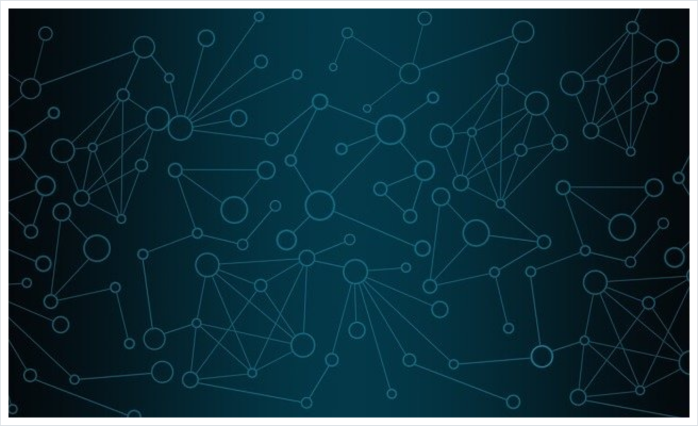
- $W \models \Diamond\varphi[\nu]$ if there is an accessibility mapping $f\colon W \to U$ such that $U \models \varphi[f \circ \nu]$;
- $W \models \Box\varphi[\nu]$ if for any accessibility mappings $f\colon W \to U$, we have that $U \models \varphi[f \circ \nu]$.

## Example

Consider the category of groups under homomorphisms. Then, for any group $\langle G, \cdot, e\rangle$, we have that

$$G \models \forall x \, \Diamond(x = e)$$

# Modal graph theory



We work under inclusions.

We work under inclusions.

**Theorem**

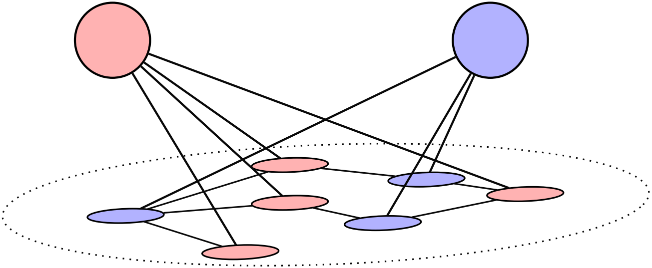
*2-colorability is expressible in modal graph theory.*

We work under inclusions.

**Theorem**

*2-colorability is expressible in modal graph theory.*

**Proof.**



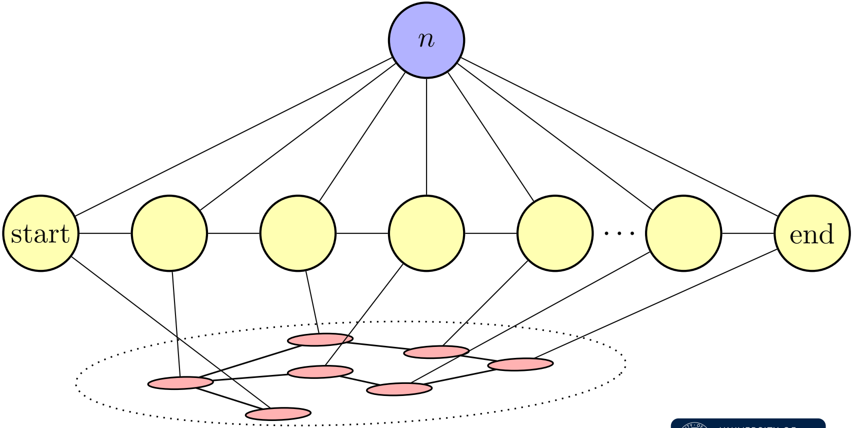
### Theorem

*Finiteness is expressible in modal graph theory.*

Theorem

*Finiteness is expressible in modal graph theory.*

Proof.



Much more is expressible.

Much more is expressible.

- sizes $\aleph_1$, $\aleph_2$, $\ldots$,

Much more is expressible.

- sizes $\aleph_1$, $\aleph_2$, $\ldots$,
- sizes $\aleph_\omega$, $\beth_\omega$,

Much more is expressible.

- sizes $\aleph_1$, $\aleph_2$, $\ldots$,
- sizes $\aleph_\omega$, $\beth_\omega$,
- sizes of the least $\beth$-fixed point, $\beth$-hyper-fixed point, $\ldots$,

Much more is expressible.

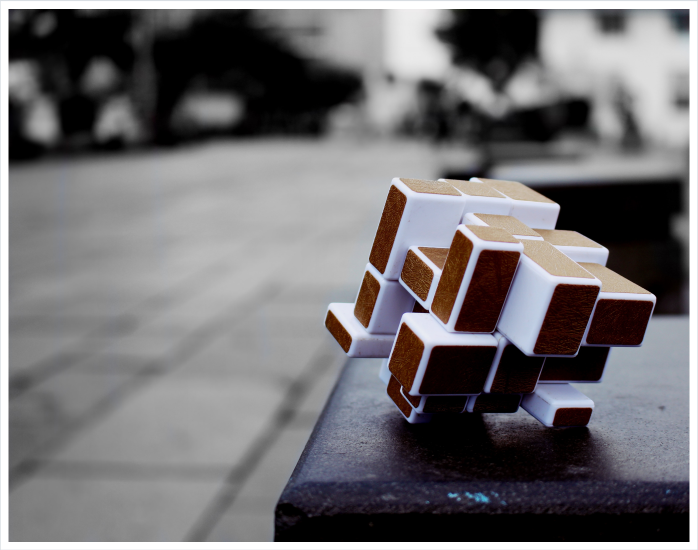
- sizes $\aleph_1$, $\aleph_2$, $\ldots$,
- sizes $\aleph_\omega$, $\beth_\omega$,
- sizes of the least $\beth$-fixed point, $\beth$-hyper-fixed point, $\ldots$,
- Much more.

Much more is expressible.

- sizes $\aleph_1$, $\aleph_2$, $\ldots$,
- sizes $\aleph_\omega$, $\beth_\omega$,
- sizes of the least $\beth$-fixed point, $\beth$-hyper-fixed point, $\ldots$,
- Much more.

A large fragment of set-theoretic truth is interpretable in modal graph theory.

# Modal group theory



Let us work under embeddings.

Let us work under embeddings.

**Theorem**

*The property of being an element of a group generated by another element of that group is expressible in the modal language of group theory.*

Let us work under embeddings.

**Theorem**

*The property of being an element of a group generated by another element of that group is expressible in the modal language of group theory.*

**Proof.**

We claim that $y \in \langle x \rangle$ if and only if

$$\Box \forall t \, (tx = xt \to ty = yt) \, .$$

Let us work under embeddings.

**Theorem**

*The property of being an element of a group generated by another element of that group is expressible in the modal language of group theory.*

**Proof.**

We claim that $y \in \langle x \rangle$ if and only if

$$\Box \forall t \, (tx = xt \to ty = yt) \, .$$

$(\to)$:

Let us work under embeddings.

**Theorem**

*The property of being an element of a group generated by another element of that group is expressible in the modal language of group theory.*

**Proof.**

We claim that $y \in \langle x \rangle$ if and only if

$$\Box \forall t \, (tx = xt \to ty = yt) \, .$$

$(\to)$: easy.

Let us work under embeddings.

**Theorem**

*The property of being an element of a group generated by another element of that group is expressible in the modal language of group theory.*

**Proof.**

We claim that $y \in \langle x \rangle$ if and only if

$$\Box \forall t \, (tx = xt \to ty = yt) \, .$$

$(\to)$: easy.

$(\leftarrow)$:

Let us work under embeddings.

### Theorem

*The property of being an element of a group generated by another element of that group is expressible in the modal language of group theory.*

### Proof.

We claim that $y \in \langle x \rangle$ if and only if

$$\Box \forall t \, (tx = xt \to ty = yt) \, .$$

$(\rightarrow)$: easy.
$(\leftarrow)$: Assume $G$ has elements $x$ and $y$ with the property that necessarily for any $t$, $xt = tx$ entails $yt = ty$. $\square$

Consider an HNN extension, relative to the identity automorphism on the group generated by $x$.

Consider an HNN extension, relative to the identity automorphism on the group generated by $x$.

$$G^* = \langle G, s \mid s\gamma = \gamma s \text{ for } \gamma \in \langle x \rangle \rangle .$$

Consider an HNN extension, relative to the identity automorphism on the group generated by $x$.

$$G^* = \langle G, s \mid s\gamma = \gamma s \text{ for } \gamma \in \langle x \rangle \rangle .$$

Since it is HNN, we have an embedding $g \mapsto g$ from $G$ to $G^*$.

Consider an HNN extension, relative to the identity automorphism on the group generated by $x$.

$$G^* = \langle G, s \mid s\gamma = \gamma s \text{ for } \gamma \in \langle x \rangle \rangle .$$

Since it is HNN, we have an embedding $g \mapsto g$ from $G$ to $G^*$. Because necessarily for any $t$, $xt = tx$ entails $yt = ty$, we have that $y$ commutes with $s$.

Consider an HNN extension, relative to the identity automorphism on the group generated by $x$.

$$G^* = \langle G, s \mid s\gamma = \gamma s \text{ for } \gamma \in \langle x \rangle \rangle .$$

Since it is HNN, we have an embedding $g \mapsto g$ from $G$ to $G^*$. Because necessarily for any $t$, $xt = tx$ entails $yt = ty$, we have that $y$ commutes with $s$.

$$sys^{-1}y^{-1} = e,$$

Consider an HNN extension, relative to the identity automorphism on the group generated by $x$.

$$G^* = \langle G, s \mid s\gamma = \gamma s \text{ for } \gamma \in \langle x \rangle \rangle .$$

Since it is HNN, we have an embedding $g \mapsto g$ from $G$ to $G^*$. Because necessarily for any $t$, $xt = tx$ entails $yt = ty$, we have that $y$ commutes with $s$.

$$sys^{-1}y^{-1} = e,$$

But any element of an HNN extension that can be presented as a non-trivial reduced word, cannot be the identity in the HNN extension.

Consider an HNN extension, relative to the identity automorphism on the group generated by $x$.

$$G^* = \langle G, s \mid s\gamma = \gamma s \text{ for } \gamma \in \langle x \rangle \rangle .$$

Since it is HNN, we have an embedding $g \mapsto g$ from $G$ to $G^*$. Because necessarily for any $t$, $xt = tx$ entails $yt = ty$, we have that $y$ commutes with $s$.

$$sys^{-1}y^{-1} = e,$$

But any element of an HNN extension that can be presented as a non-trivial reduced word, cannot be the identity in the HNN extension. So $y$ must be in the group generated by $x$ in $G$.

More is expressible!

More is expressible!

**Theorem**

*The property of being an element of a finite order is expressible in the modal language of group theory.*

More is expressible!

**Theorem**
*The property of being an element of a finite order is expressible in the modal language of group theory.*

It turns out that the ring of integers is interpretable in modal group theory.

More is expressible!

**Theorem**

*The property of being an element of a finite order is expressible in the modal language of group theory.*

It turns out that the ring of integers is interpretable in modal group theory. In particular, the modal theory of the trivial group is (highly) undecidable.

# Modal validities

# Modal axioms

Well-known modal axioms include:

# Modal axioms



Well-known modal axioms include:

| | |
|---|---|
| K | $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ |
| S | $\Box\varphi \to \varphi$ |
| 4 | $\Box\varphi \to \Box\Box\varphi$ |

Well-known modal axioms include:

$$
\begin{array}{rl}
\mathrm{K} & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
\mathrm{S} & \Box\varphi \to \varphi \\
4 & \Box\varphi \to \Box\Box\varphi \\
.2 & \Diamond\Box\varphi \to \Box\Diamond\varphi \\
.3 & (\Diamond\varphi \wedge \Diamond\psi) \to \Diamond\big((\varphi \wedge \Diamond\psi) \vee (\Diamond\varphi \wedge \psi)\big) \\
5 & \Diamond\Box\varphi \to \varphi.
\end{array}
$$

Well-known modal axioms include:

| | |
|---|---|
| K | $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ |
| S | $\Box\varphi \to \varphi$ |
| 4 | $\Box\varphi \to \Box\Box\varphi$ |
| .2 | $\Diamond\Box\varphi \to \Box\Diamond\varphi$ |
| .3 | $(\Diamond\varphi \wedge \Diamond\psi) \to \Diamond\big((\varphi \wedge \Diamond\psi) \vee (\Diamond\varphi \wedge \psi)\big)$ |
| 5 | $\Diamond\Box\varphi \to \varphi.$ |
| Grz | $\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$ |

Well-known modal axioms include:

$$
\begin{array}{rl}
\mathrm{K} & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
\mathrm{S} & \Box\varphi \to \varphi \\
4 & \Box\varphi \to \Box\Box\varphi \\
.2 & \Diamond\Box\varphi \to \Box\Diamond\varphi \\
.3 & (\Diamond\varphi \wedge \Diamond\psi) \to \Diamond\big((\varphi \wedge \Diamond\psi) \vee (\Diamond\varphi \wedge \psi)\big) \\
5 & \Diamond\Box\varphi \to \varphi. \\
\mathrm{Grz} & \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi
\end{array}
$$

### Definition

A modal assertion $\varphi(p_0, \ldots, p_n)$ is valid at a world for an allowed language if all substitution instances $\varphi(\psi_0, \ldots, \psi_n)$ arising for $\psi_i$ in that language are true at that world.

# Seeking modal validities—you can do it!

M $\quad \Box\Diamond\varphi \to \Diamond\Box\varphi.$

# Seeking modal validities—you can do it!



M $\qquad \Box\Diamond\varphi \to \Diamond\Box\varphi.$

Valid for graph theory under inclusions?

M $\Box\Diamond\varphi \to \Diamond\Box\varphi.$

Valid for graph theory under inclusions?

No.

M $\qquad \Box\Diamond\varphi \to \Diamond\Box\varphi.$

Valid for graph theory under inclusions?

No. It is necessarily possible that the diameter is two. You can extend any graph to make it true. But this can also be destroyed by adding isolated points.

M $\quad \Box\Diamond\varphi \to \Diamond\Box\varphi.$

Valid for graph theory under inclusions?

No. It is necessarily possible that the diameter is two. You can extend any graph to make it true. But this can also be destroyed by adding isolated points.

$\Diamond\Box$Triv $\quad \Diamond\Box(\varphi \leftrightarrow \Box\varphi).$

$$\text{M} \qquad \Box\Diamond\varphi \to \Diamond\Box\varphi.$$

Valid for graph theory under inclusions?

No. It is necessarily possible that the diameter is two. You can extend any graph to make it true. But this can also be destroyed by adding isolated points.

$$\Diamond\Box\text{Triv} \qquad \Diamond\Box(\varphi \leftrightarrow \Box\varphi).$$

Valid for sets under surjections?

M $\quad \Box\Diamond\varphi \to \Diamond\Box\varphi.$

Valid for graph theory under inclusions?

No. It is necessarily possible that the diameter is two. You can extend any graph to make it true. But this can also be destroyed by adding isolated points.

$\Diamond\Box$Triv $\quad \Diamond\Box(\varphi \leftrightarrow \Box\varphi).$

Valid for sets under surjections?

Yes.

$$\text{M} \qquad \Box\Diamond\varphi \to \Diamond\Box\varphi.$$

Valid for graph theory under inclusions?

No. It is necessarily possible that the diameter is two. You can extend any graph to make it true. But this can also be destroyed by adding isolated points.

$$\Diamond\Box\text{Triv} \qquad \Diamond\Box(\varphi \leftrightarrow \Box\varphi).$$

Valid for sets under surjections?

Yes. The condition $\varphi \leftrightarrow \Box\varphi$ is called *modality trivialization*.

M $\quad \Box\Diamond\varphi \to \Diamond\Box\varphi.$

Valid for graph theory under inclusions?

No. It is necessarily possible that the diameter is two. You can extend any graph to make it true. But this can also be destroyed by adding isolated points.

$\Diamond\Box$Triv $\quad \Diamond\Box(\varphi \leftrightarrow \Box\varphi).$

Valid for sets under surjections?

Yes. The condition $\varphi \leftrightarrow \Box\varphi$ is called *modality trivialization*. How about under inclusions?

M $\Box\Diamond\varphi \to \Diamond\Box\varphi$.

Valid for graph theory under inclusions?

No. It is necessarily possible that the diameter is two. You can extend any graph to make it true. But this can also be destroyed by adding isolated points.

$\Diamond\Box$Triv $\Diamond\Box(\varphi \leftrightarrow \Box\varphi)$.

Valid for sets under surjections?

Yes. The condition $\varphi \leftrightarrow \Box\varphi$ is called *modality trivialization.*
How about under inclusions?
Hint: Every formula is equivalent in that category to a Boolean combination of the equality and non-equality patterns of finitely many variables and first-order sentences expressing that there are at least $n$ elements.

Let us list important modal theories that arise from the aforementioned well-established modal principles. Here, "+" forces the resulting theory to be *normal*, in particular closed under modus ponens and necessitation.

Let us list important modal theories that arise from the aforementioned well-established modal principles. Here, "+" forces the resulting theory to be *normal*, in particular closed under modus ponens and necessitation.

$$
\begin{array}{rcl}
\mathrm{Triv} & = & \mathrm{Triv} + 4 = \mathrm{S5} + \mathrm{Grz} \\
\mathrm{S5} & = & \mathrm{S4} + 5 \\
\mathrm{S4.3} & = & \mathrm{S4} + .3 \\
\mathrm{S4.2} & = & \mathrm{S4} + .2 \\
\mathrm{Grz.3} & = & \mathrm{S4.3} + \mathrm{Grz} \\
\mathrm{Grz.2} & = & \mathrm{S4.2} + \mathrm{Grz} \\
\mathrm{Grz} & = & \mathrm{K} + \mathrm{Grz} = \mathrm{S4} + \mathrm{Grz} \\
\mathrm{S4} & = & \mathrm{K} + 4
\end{array}
$$

# Common modal theories

Let us list important modal theories that arise from the aforementioned well-established modal principles. Here, "+" forces the resulting theory to be *normal*, in particular closed under modus ponens and necessitation.

$$
\begin{aligned}
\mathrm{Triv} &= \mathrm{Triv} + 4 = \mathrm{S5} + \mathrm{Grz} \\
\mathrm{S5} &= \mathrm{S4} + 5 \\
\mathrm{S4.3} &= \mathrm{S4} + .3 \\
\mathrm{S4.2} &= \mathrm{S4} + .2 \\
\mathrm{Grz.3} &= \mathrm{S4.3} + \mathrm{Grz} \\
\mathrm{Grz.2} &= \mathrm{S4.2} + \mathrm{Grz} \\
\mathrm{Grz} &= \mathrm{K} + \mathrm{Grz} = \mathrm{S4} + \mathrm{Grz} \\
\mathrm{S4} &= \mathrm{K} + 4
\end{aligned}
$$

$$
\begin{tikzcd}
& \mathrm{Triv} \arrow[dl] \arrow[dr] & \\
\mathrm{S5} \arrow[dr] & & \mathrm{Grz.3} \arrow[dl] \arrow[d] \\
& \mathrm{S4.3} \arrow[d] & \mathrm{Grz.2} \arrow[dl] \arrow[d] \\
& \mathrm{S4.2} \arrow[d] & \mathrm{Grz} \arrow[dl] \\
& \mathrm{S4} &
\end{tikzcd}
$$

# Easy lower bounds



- S4 is universally valid in Kripke categories.

# Easy lower bounds



- S4 is universally valid in Kripke categories.
- If worlds are convergent, then S4.2 is valid for sentences.

# Easy lower bounds



- S4 is universally valid in Kripke categories.
- If worlds are convergent, then S4.2 is valid for sentences.
- If the the diagrams in question moreover commute, then S4.2 is also valid for formulas.

# Easy lower bounds



- S4 is universally valid in Kripke categories.
- If worlds are convergent, then S4.2 is valid for sentences.
- If the the diagrams in question moreover commute, then S4.2 is also valid for formulas.
- In $\mathrm{Mod}(T)$ under embeddings, the converse also holds.

# Easy lower bounds



- S4 is universally valid in Kripke categories.
- If worlds are convergent, then S4.2 is valid for sentences.
- If the the diagrams in question moreover commute, then S4.2 is also valid for formulas.
- In $\mathrm{Mod}(T)$ under embeddings, the converse also holds.
- S4.3 is valid just in case there are no two independent buttons—possibly necessary statements that one can make true separately, without interfering with one another.

Definition

An assertion $\varphi$ is penultimate if $\neg\varphi \wedge \Diamond\varphi \wedge \Box(\varphi \to \Box\varphi)$.

### Definition

An assertion $\varphi$ is penultimate if $\neg\varphi \wedge \Diamond\varphi \wedge \Box(\varphi \to \Box\varphi)$.

### Theorem

*If* S4.2 *is valid, then* Grz *is valid if and only if for each* $\varphi$, *either* $\varphi$ *or* $\neg\varphi$ *is necessary or possibly penultimate on a cone.*

# Penultimacy

## Definition

An assertion $\varphi$ is penultimate if $\neg\varphi \wedge \Diamond\varphi \wedge \Box(\varphi \to \Box\varphi)$.

## Theorem

*If* S4.2 *is valid, then* Grz *is valid if and only if for each $\varphi$, either $\varphi$ or $\neg\varphi$ is necessary or possibly penultimate on a cone.*

## Key to proof.

Observe that under S4.2, Grz becomes symmetric. $\square$

### Definition

An assertion $\varphi$ is penultimate if $\neg\varphi \wedge \Diamond\varphi \wedge \Box(\varphi \to \Box\varphi)$.

### Theorem

*If* S4.2 *is valid, then* Grz *is valid if and only if for each* $\varphi$*, either* $\varphi$ *or* $\neg\varphi$ *is necessary or possibly penultimate on a cone.*

### Key to proof.

Observe that under S4.2, Grz becomes symmetric. $\square$

### Is Grz valid for sets under surjections?

### Definition

An assertion $\varphi$ is penultimate if $\neg\varphi \wedge \Diamond\varphi \wedge \Box(\varphi \to \Box\varphi)$.

### Theorem

*If* S4.2 *is valid, then* Grz *is valid if and only if for each* $\varphi$*, either* $\varphi$ *or* $\neg\varphi$ *is necessary or possibly penultimate on a cone.*

### Key to proof.

Observe that under S4.2, Grz becomes symmetric. $\square$

### Is Grz valid for sets under surjections?

Yes.

### Definition

An assertion $\varphi$ is penultimate if $\neg\varphi \wedge \Diamond\varphi \wedge \Box(\varphi \to \Box\varphi)$.

### Theorem

*If* S4.2 *is valid, then* Grz *is valid if and only if for each* $\varphi$*, either* $\varphi$ *or* $\neg\varphi$ *is necessary or possibly penultimate on a cone.*

### Key to proof.

Observe that under S4.2, Grz becomes symmetric. $\square$

### Is Grz valid for sets under surjections?

Yes. Show that .2 is valid and use the theorem.

**Theorem**

*A world that admits arbitrarily many independent buttons has its propositional modal validities contained within* Grz.2.

# The case of buttons but no switches

**Theorem**

*A world that admits arbitrarily many independent buttons has its propositional modal validities contained within* Grz.2.

**Proof sketch.**

- It is known that Grz.2 is characterized by finite partial orders with the largest element.

# The case of buttons but no switches



**Theorem**

*A world that admits arbitrarily many independent buttons has its propositional modal validities contained within* Grz.2*.*

**Proof sketch.**

- It is known that Grz.2 is characterized by finite partial orders with the largest element.
- Goal: show that Grz.2 is characterized by finite lattices/Boolean algebras.

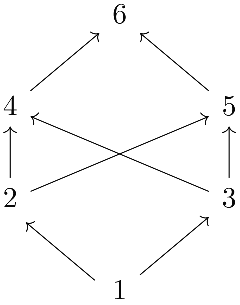
$\square$

- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.

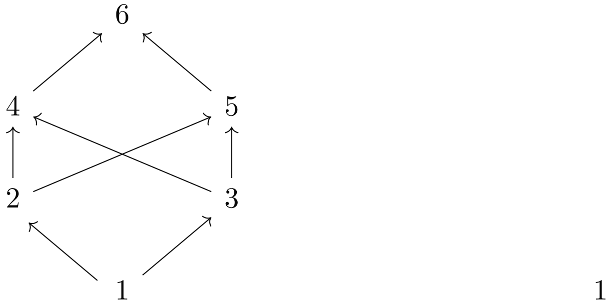
# The case of buttons but no switches

- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.

- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.



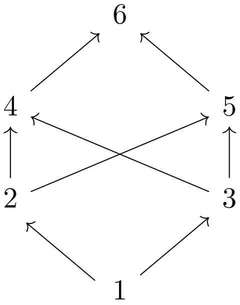
- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.

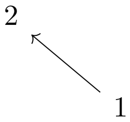


# The case of buttons but no switches



- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.

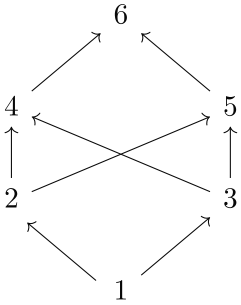


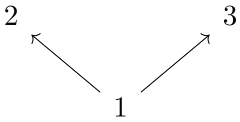



# The case of buttons but no switches



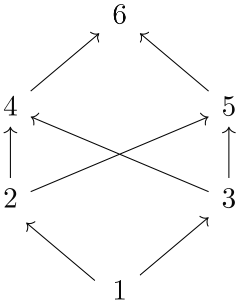
- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.

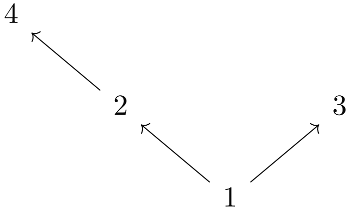


# The case of buttons but no switches

- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.

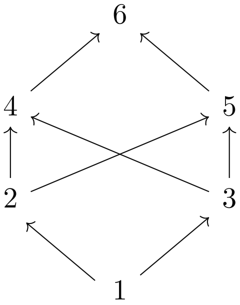


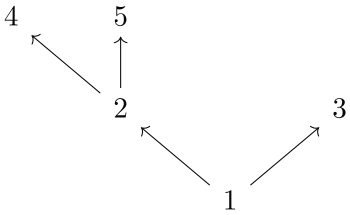



# The case of buttons but no switches

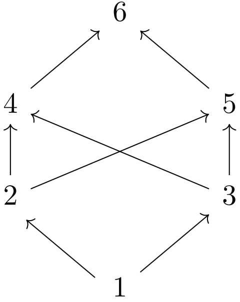


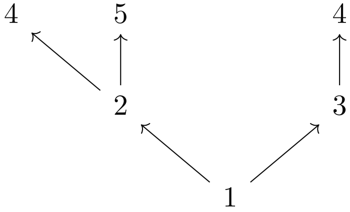
- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.



- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.

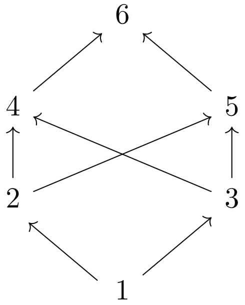


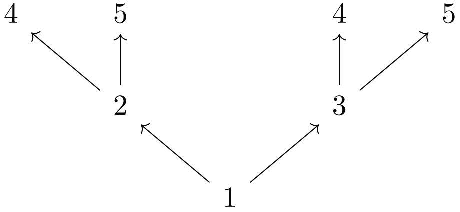



# The case of buttons but no switches



- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.

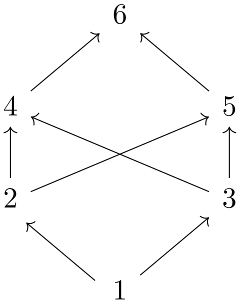


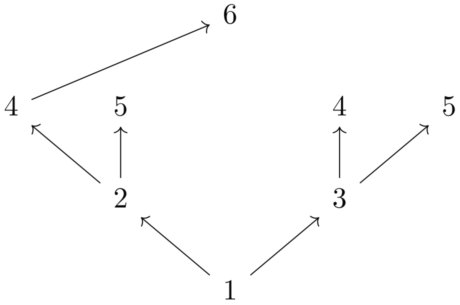



# The case of buttons but no switches



- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.

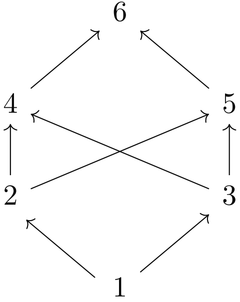


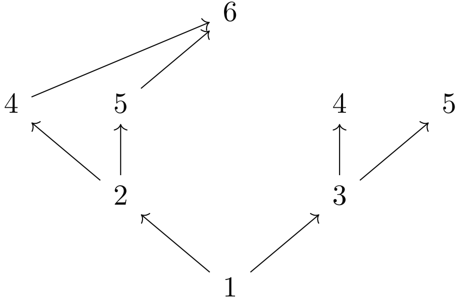



# The case of buttons but no switches



- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.

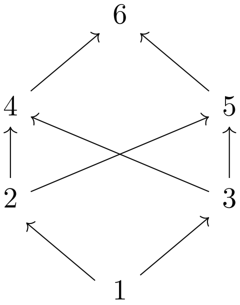


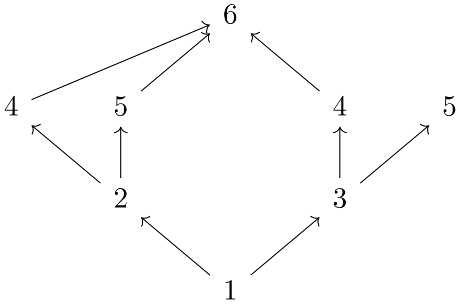



# The case of buttons but no switches



- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.

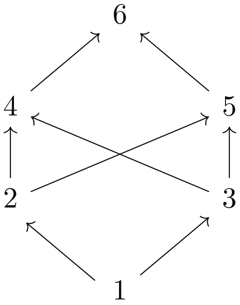


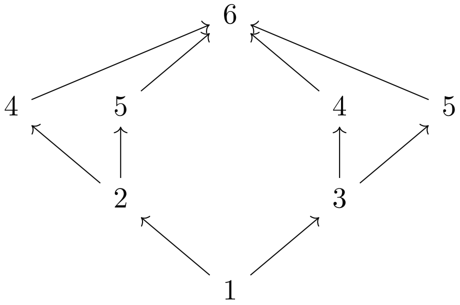



# The case of buttons but no switches



- If Grz.2 $\not\models \varphi$, then there is a finite partially ordered Kripke model with the largest element $M_0$ wherein $\varphi$ fails.
- Use partial tree unraveling to construct a baled tree $M$ bisimilar to $M_0$.



- provide an assignment of the propositional variables $p_i$ to assertions $\psi_{p_i}$ in the modal language, so that $(M, w) \models \varphi(p_0, \ldots, p_n)$ just in case $W \models \varphi(\psi_{p_0}, \ldots, \psi_{p_n})$.

- provide an assignment of the propositional variables $p_i$ to assertions $\psi_{p_i}$ in the modal language, so that $(M,w) \models \varphi(p_0,\ldots,p_n)$ just in case $W \models \varphi(\psi_{p_0},\ldots,\psi_{p_n})$.
- $\varphi$ fails at $w$, so $W \models \neg\varphi(\psi_{p_0},\ldots,\psi_{p_n})$, and Grz.2 cannot be valid at $W$.

- provide an assignment of the propositional variables $p_i$ to assertions $\psi_{p_i}$ in the modal language, so that $(M, w) \models \varphi(p_0, \ldots, p_n)$ just in case $W \models \varphi(\psi_{p_0}, \ldots, \psi_{p_n})$.
- $\varphi$ fails at $w$, so $W \models \neg\varphi(\psi_{p_0}, \ldots, \psi_{p_n})$, and Grz.2 cannot be valid at $W$.

### Theorem

*Under surjections, any infinite set validates exactly* Grz.2 *for assertions with parameters.*

# Exact modal validities for the category of sets



| | Propositional modal validities | | | | |
|---|---|---|---|---|---|
| | Empty world | Finite worlds of size $n > 0$ | | Infinite worlds | |
| Category Set with: | — | Sentential | Formulaic | Sentential | Formulaic |
| Homomorphisms | Lollipop | S5 | $\text{Prepartition}_n$ | S5 | S4.2 |
| Epimorphisms | Triv | $\text{Grz.3J}_n$ | $\text{Partition}_n$ | Grz.3 | Grz.2 |
| Monomorphisms | Grz.3 | Grz.3 | Grz.3 | Triv | Triv |
| Inclusions | Grz.3 | Grz.3 | Grz.3 | Triv | Triv |
| Isomorphisms | Triv | Triv | Triv | Triv | Triv |
| Identities | Triv | Triv | Triv | Triv | Triv |

# Thank you!

@WAWoloszyn www.woloszyn.org

Other recent work on modal logic and modal model theory and references:


[Ada22] Sam Adam-Day. "Bisimulations of potentialist systems". In: *Mathematics arXiv* (2022). Under review.

[Ham18] Joel David Hamkins. "The modal logic of arithmetic potentialism and the universal algorithm". In: *ArXiv e-prints* (2018). Under review, pp. 1–35. arXiv: `1801.04599 [math.LO]`. URL: `http://wp.me/p5M0LV-1Dh`.

[HL08] Joel Hamkins and Benedikt Löwe. "The modal logic of forcing". In: *Transactions of the American Mathematical Society* 360.4 (2008), pp. 1793–1817.

[HL19] Joel David Hamkins and Øystein Linnebo. "The modal logic of set-theoretic potentialism and the potentialist maximality principles". In: *Review of Symbolic Logic* (October 2019). DOI: `10.1017/S1755020318000242`. arXiv: `1708.01644 [math.LO]`. URL: `http://wp.me/p5M0LV-1zC`.

[HW20] Joel David Hamkins and Wojciech Aleksander Wołoszyn. "Modal model theory". In: *Mathematics arXiv* (2020). Under review. arXiv: `2009.09394 [math.LO]`.

[SS20] Denis I. Saveliev and Ilya Shapirovsky. "On Modal Logics of Model-Theoretic Relations". In: *Stud Logica* 108 (2020), pp. 989–1017.