



The tension between large cardinals and resurrection



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The *resurrection axiom* with real parameters $\text{RA}(\mathbb{R})$ is the assertion that, for every formula $\varphi(x)$ and $r \in \mathbb{R}$, if $\varphi(r)$ is true in V , then $\varphi(r)$ is necessarily forceable.

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The *necessary resurrection axiom* with real parameters $\square\text{RA}(\mathbb{R})$ is the assertion that every poset \mathbb{P} forces $\text{RA}(\mathbb{R}^{V^{\mathbb{P}}})$.

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The intersection of a set of grounds contains a ground.

A cardinal κ is *extendible* if for every ordinal λ , there is an ordinal $\gamma > \lambda$ and an elementary embedding $j: V_\lambda \rightarrow V_\gamma$ with a critical point equal to κ , which is to say that for any $x \in V_\lambda$, the cardinality of x is invariant under j just in case its less than κ .

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Theorem (Usuba)

If there is an extendible cardinal, then the mantle is a ground.

Corollary

If the mantle is a ground, then the necessary resurrection axiom with real parameters $\square\text{RA}(\mathbb{R})$ fails.

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It is consistent that Usuba's theorem fails in V_κ , where κ is the least extendible cardinal.

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The assertion “ κ is an extendible cardinal” is Π_3 -expressible in the language of set theory.

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Observation

The assertion “ κ is an extendible cardinal” is Π_3 -expressible in the language of set theory.

Prediction

Usuba’s theorem is about complexity strength as opposed to consistency strength.

For a class of assertions Γ , the *necessary Γ -resurrection axiom* with real parameters $\square\text{RA}_\Gamma(\mathbb{R})$ is the necessary resurrection axiom with real parameters qualified to formulas $\varphi(x)$ from the class Γ .

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Theorem (Woodin)

If there is a proper class of Woodin cardinals, then the necessary Σ_2 -resurrection axiom with real parameters $\square\text{RA}_{\Sigma_2}(\mathbb{R})$ holds.

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Theorem

If the mantle is a ground, then the necessary Π_3 -resurrection axiom with real parameters $\square\text{RA}_{\Pi_3}(\mathbb{R})$ fails.

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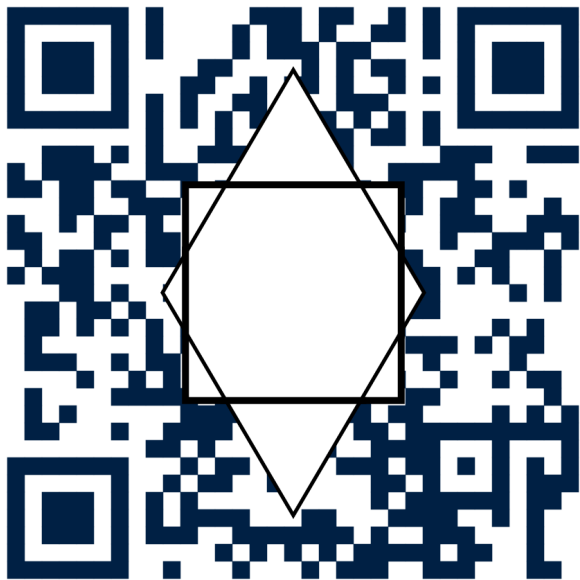
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- ▶ connections with $V = \text{Ultimate } L$.

Thank you!



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