The tension between large cardinals and resurrection

Wojciech Aleksander Wołoszyn

@WAWoloszyn www.woloszyn.org

Mathematical Institute
University of Oxford

St Hilda's College
Oxford

Set Theory in the United Kingdom
7th March 2023

Follow along!

Suppose $\varphi(x)$ is a formula in the language of set theory with a free variable x, and that s is a set.

Suppose $\varphi(x)$ is a formula in the language of set theory with a free variable x, and that s is a set. A sentence $\varphi(s)$ is

Resurrection axioms

Suppose $\varphi(x)$ is a formula in the language of set theory with a free variable x, and that s is a set. A sentence $\varphi(s)$ is

- *forceable* if there is a poset $\mathbb{P}$ which forces $\varphi(s)$;

Resurrection axioms

Suppose $\varphi(x)$ is a formula in the language of set theory with a free variable x, and that s is a set. A sentence $\varphi(s)$ is

- *forceable* if there is a poset $\mathbb{P}$ which forces $\varphi(s)$;
- *necessary* if every poset $\mathbb{P}$ forces $\varphi(s)$.

Resurrection axioms

Suppose $\varphi(x)$ is a formula in the language of set theory with a free variable x, and that s is a set. A sentence $\varphi(s)$ is

- *forceable* if there is a poset $\mathbb{P}$ which forces $\varphi(s)$;
- *necessary* if every poset $\mathbb{P}$ forces $\varphi(s)$.

The *resurrection axiom* with real parameters $\mathrm{RA}(\mathbb{R})$ is the assertion that, for every formula $\varphi(x)$ and $r \in \mathbb{R}$, if $\varphi(r)$ is true in V, then $\varphi(r)$ is necessarily forceable.

Suppose $\varphi(x)$ is a formula in the language of set theory with a free variable x, and that s is a set. A sentence $\varphi(s)$ is

- *forceable* if there is a poset $\mathbb{P}$ which forces $\varphi(s)$;
- *necessary* if every poset $\mathbb{P}$ forces $\varphi(s)$.

The *resurrection axiom* with real parameters $\mathrm{RA}(\mathbb{R})$ is the assertion that, for every formula $\varphi(x)$ and $r \in \mathbb{R}$, if $\varphi(r)$ is true in V, then $\varphi(r)$ is necessarily forceable.

The *necessary resurrection axiom* with real parameters $\Box\mathrm{RA}(\mathbb{R})$ is the assertion that every poset $\mathbb{P}$ forces $\mathrm{RA}(\mathbb{R}^{V^{\mathbb{P}}})$.

Set-theoretic geology is concerned with inner models that admit a forcing extension to V.

Set-theoretic geology is concerned with inner models that admit a forcing extension to V.

- A *ground* is an inner model with a poset $\mathbb{P} \in M$ and an $(M, \mathbb{P})$-generic $G \subseteq \mathbb{P}$ such that $M[G] = V$.

Set-theoretic geology is concerned with inner models that admit a forcing extension to V.

- A *ground* is an inner model with a poset $\mathbb{P} \in M$ and an $(M, \mathbb{P})$-generic $G \subseteq \mathbb{P}$ such that $M[G] = V$.
- The *mantle* $\mathbb{M}$ is the intersection of all grounds of V.

Set-theoretic geology is concerned with inner models that admit a forcing extension to V.

- A *ground* is an inner model with a poset $\mathbb{P} \in M$ and an $(M, \mathbb{P})$-generic $G \subseteq \mathbb{P}$ such that $M[G] = V$.
- The *mantle* $\mathbb{M}$ is the intersection of all grounds of V.

Theorem (Usuba)

The mantle is a model of ZFC.

Set-theoretic geology is concerned with inner models that admit a forcing extension to V.

- A *ground* is an inner model with a poset $\mathbb{P} \in M$ and an $(M, \mathbb{P})$-generic $G \subseteq \mathbb{P}$ such that $M[G] = V$.
- The *mantle* $\mathbb{M}$ is the intersection of all grounds of V.

Theorem (Usuba)

The mantle is a model of ZFC.

Theorem (Usuba)

The intersection of a set of grounds contains a ground.

A cardinal κ is *extendible* if for every ordinal λ, there is an ordinal $\gamma > \lambda$ and an elementary embedding $j\colon V_\lambda \to V_\gamma$ with a critical point equal to κ, which is to say that for any $x \in V_\lambda$, the cardinality of x is invariant under j just in case its less than κ.

A cardinal κ is *extendible* if for every ordinal λ, there is an ordinal $\gamma > \lambda$ and an elementary embedding $j\colon V_\lambda \to V_\gamma$ with a critical point equal to κ, which is to say that for any $x \in V_\lambda$, the cardinality of x is invariant under j just in case its less than κ.

Theorem (Usuba)

If there is an extendible cardinal, then the mantle is a ground.

A cardinal κ is *extendible* if for every ordinal λ, there is an ordinal $\gamma > \lambda$ and an elementary embedding $j\colon V_\lambda \to V_\gamma$ with a critical point equal to κ, which is to say that for any $x \in V_\lambda$, the cardinality of x is invariant under j just in case its less than κ.

Theorem (Usuba)

If there is an extendible cardinal, then the mantle is a ground.

Corollary

If the mantle is a ground, then the necessary resurrection axiom with real parameters $\Box\mathrm{RA}(\mathbb{R})$ *fails.*

Optimality of an extendible?

Usuba asked whether the large cardinal assumption in his theorem can be weakened.

Optimality of an extendible?

Usuba asked whether the large cardinal assumption in his theorem can be weakened.

And it was answered positively.

Usuba asked whether the large cardinal assumption in his theorem can be weakened.

And it was answered positively.

Theorem (Goldberg)

It is consistent that Usuba's theorem fails in V_κ, where κ is the least extendible cardinal.

Usuba asked whether the large cardinal assumption in his theorem can be weakened.

And it was answered positively.

Theorem (Goldberg)

It is consistent that Usuba's theorem fails in V_κ, where κ is the least extendible cardinal.

But does that fully answer the question?

The existence of an extendible cardinal is not only one of the strongest axioms of infinity but also a statement with an unusually high order of complexity for such an axiom.

The existence of an extendible cardinal is not only one of the strongest axioms of infinity but also a statement with an unusually high order of complexity for such an axiom.

Observation

The assertion "κ is an extendible cardinal" is Π_3-expressible in the language of set theory.

The existence of an extendible cardinal is not only one of the strongest axioms of infinity but also a statement with an unusually high order of complexity for such an axiom.

Observation

The assertion "κ is an extendible cardinal" is Π_3-expressible in the language of set theory.

Prediction

Usuba's theorem is about complexity strength as opposed to consistency strength.

For a class of assertions Γ, the *necessary* Γ-*resurrection axiom* with real parameters $\Box\mathrm{RA}_\Gamma(\mathbb{R})$ is the necessary resurrection axiom with real parameters qualified to formulas $\varphi(x)$ from the class Γ.

For a class of assertions Γ, the *necessary* Γ*-resurrection axiom* with real parameters $\Box\mathrm{RA}_\Gamma(\mathbb{R})$ is the necessary resurrection axiom with real parameters qualified to formulas $\varphi(x)$ from the class Γ.

Theorem (Woodin)

If there is a proper class of Woodin cardinals, then the necessary Σ_2*-resurrection axiom with real parameters* $\Box\mathrm{RA}_{\Sigma_2}(\mathbb{R})$ *holds.*

For a class of assertions Γ, the *necessary Γ-resurrection axiom with real parameters* $\Box\mathrm{RA}_\Gamma(\mathbb{R})$ is the necessary resurrection axiom with real parameters qualified to formulas $\varphi(x)$ from the class Γ.

Theorem (Woodin)

If there is a proper class of Woodin cardinals, then the necessary Σ_2-resurrection axiom with real parameters $\Box\mathrm{RA}_{\Sigma_2}(\mathbb{R})$ holds.

Theorem

If the mantle is a ground, then the necessary Π_3-resurrection axiom with real parameters $\Box\mathrm{RA}_{\Pi_3}(\mathbb{R})$ fails.

Current work and my DPhil project

So... so far in this DPhil project...

So... so far in this DPhil project...

We verified that necessary Π_3 resurrection with real parameters $\Box\mathrm{RA}_{\Pi_3}(\mathbb{R})$ is false, if the mantle is a ground.

So... so far in this DPhil project...

We verified that necessary Π_3 resurrection with real parameters $\Box\mathrm{RA}_{\Pi_3}(\mathbb{R})$ is false, if the mantle is a ground.

And current work in progress concerns:

So... so far in this DPhil project...

We verified that necessary Π_3 resurrection with real parameters $\Box\mathrm{RA}_{\Pi_3}(\mathbb{R})$ is false, if the mantle is a ground.

And current work in progress concerns:

- consistency of $\Box\mathrm{RA}(\mathbb{R})$ with *extremely* large cardinals;

So... so far in this DPhil project...

We verified that necessary Π_3 resurrection with real parameters $\Box\mathrm{RA}_{\Pi_3}(\mathbb{R})$ is false, if the mantle is a ground.

And current work in progress concerns:

- consistency of $\Box\mathrm{RA}(\mathbb{R})$ with *extremely* large cardinals;
- refuting $\Box\mathrm{RA}_{\Sigma_3}(\mathbb{R})$ and $\Box\mathrm{RA}_{\Pi_2}(\mathbb{R})$ from large cardinal assumptions;

So... so far in this DPhil project...

We verified that necessary Π_3 resurrection with real parameters $\Box\mathrm{RA}_{\Pi_3}(\mathbb{R})$ is false, if the mantle is a ground.

And current work in progress concerns:

- consistency of $\Box\mathrm{RA}(\mathbb{R})$ with *extremely* large cardinals;
- refuting $\Box\mathrm{RA}_{\Sigma_3}(\mathbb{R})$ and $\Box\mathrm{RA}_{\Pi_2}(\mathbb{R})$ from large cardinal assumptions;
- connections with $V = \text{Ultimate } L$.

Thank you!

@WAWoloszyn www.woloszyn.org

[FHR15] Gunter Fuchs, Joel David Hamkins, and Jonas Reitz. "Set-theoretic geology". In: *Annals of Pure and Applied Logic* 166.4 (2015), pp. 464–501. ISSN: 0168-0072. DOI: `10.1016/j.apal.2014.11.004`. arXiv: `1107.4776 [math.LO]`. URL: `http://jdh.hamkins.org/set-theoreticgeology`.

[Gol21] Gabriel Goldberg. *Usuba's extendible cardinal.* 2021. URL: `https://arxiv.org/abs/2108.06903`.

[Lar04] Paul B Larson. *The stationary tower : notes on a course given by W. Hugh Woodin.* eng. University lecture series (Providence, R.I.) ; 32. Providence, R.I.: American Mathematical Society, 2004. ISBN: 9780821836040.

[Usu19] Toshimichi Usuba. "Extendible cardinals and the mantle". In: *Archive for Mathematical Logic* 58 (February 2019). DOI: `10.1007/s00153-018-0625-4`.

[Woo21] W. Hugh Woodin. Personal communication. June 2021.