Resurrection principles

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Follow along!

Maximality principle

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so anything forceably necessary is already true.

Some examples and non-examples

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"$S \subseteq \omega_1$ is non-stationary" is forceably necessary, because we can always shoot a club through its complement.

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Either CH or $\neg$CH realizes an instance of the resurrection principle.

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With real parameters in hand, one can refer to any hereditarily countable set. Specifically, a real x HC-codes $y \in \mathrm{HC}$ if and only if $(\omega, \{ (n,m) \in \omega \times \omega \mid x(2^n 3^m) = 0 \}$ is isomorphic to $(\mathrm{TC}(\{y\}), \in)$.

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In particular, $0^{\#}$ exists.

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A subtle point is that full resurrections—$\Box\underset{\sim}{\mathrm{R}}\mathrm{P}$, $\underset{\sim}{\mathrm{R}}\mathrm{P}$, and RP—are taken as schemes of assertions, but $\Box\underset{\sim}{\mathrm{R}}\mathrm{P}_{\Sigma_2}$ is a single axiom, since we have a universal Σ_2 truth predicate, and can express it as one statement.

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If there is a proper class of Woodin cardinals, then $\Box\underset{\sim}{\mathrm{RP}}_{\Sigma_2}$ *holds.*

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A surprising corollary of Woodin's result is that (the existence of) I0 is resurrectable.

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Suppose M is a ground of V and $M[G] = V$ for some $\mathbb{P} \in M$ and $(M, \mathbb{P})$-generic G. Then, M is Σ_2-definable in V.

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Corollary

If BA *holds, then the mantle is* Σ_2*-definable in* V.

Usuba's extendible cardinal

A cardinal κ is *extendible* if for every ordinal λ, there is an ordinal $\gamma > \lambda$ and an elementary embedding $j \colon V_\lambda \to V_\gamma$ with a critical point equal to κ, which is to say that for any $x \in V_\lambda$, the cardinality of x is invariant under j just in case its less than κ.

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Theorem (Goldberg)

It is consistent that Usuba's theorem fails in V_κ*, where* κ *is the least extendible cardinal.*

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- By the Ground Definability Theorem, $\mathbb{M}$ is Σ_2-definable in V.
- Take a sufficiently large λ, so that $(\lambda^+)^{\mathbb{M}} = \lambda^+$.
- Let g be a V-generic real for $\mathrm{Coll}(\omega, \lambda^+)$.
- Take any strong limit $\gamma > \lambda$ with $V_\gamma \prec_{\Sigma_2} V$.

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Corollary

If there is an extendible cardinal, then $\Box \underset{\sim}{\mathrm{RP}}_{\Pi_3}$ fails.

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This, $\Box \underset{\sim}{\mathrm{RP}}_{\Pi_2}$, and more, is a work in progress.

Thank you!

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