Grz.2 IS COMPLETE FOR FINITE BOOLEAN ALGEBRAS

WOJCIECH ALEKSANDER WOŁOSZYN

Abstract. The article offers an accessible exposition of the Grzegorczyk axiom and provides a new characterization of the modal logic Grz.2, establishing that it is complete for the set of finite Boolean algebras. Thereby, it enhances the utility of the control statement technique of establishing upper bounds on the modal validities of a potentialist system [HL08; HL19; HW20].

1. Introduction

Despite its intimidating appearance, the Grzegorczyk axiom has played a central role in modal logic and has been extensively studied by logicians [Kra99]. In mathematical logic, it has appeared in the context of set theory [Sol76] and, more recently, in the subject of model theory and category theory [HW20; Woł]. It was discovered to be the missing principle needed to answer a longstanding unresolved question in control statement theory [Woł], cf. [Pas19] and [Woł22]. The axiom is stated as follows.

$$\text{Grz} \quad \Box(\Box(p \to \Box p) \to p) \to p.$$

It has been long known that modal logic Grz.2—the smallest normal modal logic containing the axioms Grz and .2—is characterized by directed partial orders [Kra99]. By employing the methods of [Fin74; Fin85], one can prove that finite directed partial orders suffice. Drawing upon the methods presented in [HL08], I give a further refinement and show that finite Boolean algebras are enough.

Main Theorem. *The modal logic* Grz.2 *is characterized by finite Boolean algebras. That is, a propositional modal assertion* φ *is provable in* Grz.2 *if and only if* φ *is valid in every finite Boolean algebra.*

It is immediate to show the forward direction of the main theorem. Every assertion provable in Grz.2 must be valid in every finite directed partial order, hence every finite Boolean algebra in particular. It therefore suffices to show that Grz.2 is complete with respect to the set of all finite Boolean algebras. More specifically, we shall show that if a propositional modal assertion φ is not valid in some finite Boolean algebra B, then φ is not provable in the modal logic Grz.2.

2. Demystifying the Grzegorczyk axiom

Let me start by elucidating the meaning of the Grzegorczyk axiom to the reader. It is convenient to rewrite it in the equivalent form $p \vee \Diamond(\neg p \wedge \Box(p \to \Box p))$, which one obtains by simply expanding the implications to disjuncts and applying de Morgan's laws for modal logic. Following [Woł], we call p *penultimate* and denote it by $\text{penultimate}(p)$, when $\neg p \wedge \Box(p \to \Box p)$. Penultimacy of p means that p is false but necessarily if it ever becomes true, it remains necessarily true. We are now poised to express the axiom in a form that is easy to digest:

$$\text{Grz} \quad p \vee \Diamond\, \text{penultimate}(p).$$

Because of the directedness, the accessibility relation in a frame that validates the modal logic Grz.2 is antiwellfounded [Woł]. As further worlds are accessed, the truth-value of any propositional variable will eventually stabilize, and no matter how we proceed, there will always be a maximal world that was wrong about the ultimate truth-value of the variable. Formally, Grz.2 proves that $\text{penultimate}(p)$ implies $\Box\Diamond p$. In this setting, therefore each propositional variable is truly possibly penultimate. It is worth emphasizing, however, that it is not true in general. There exist frames that validate the modal logic Grz and yet have a node for which $\Diamond p \vee \Diamond \neg p$ is satisfiable. See figure below for an example of a frame a world based on which can have the value of a propositional variable potentially flip indefinitely, never stabilizing and without ever attaining its ultimate truth-value.

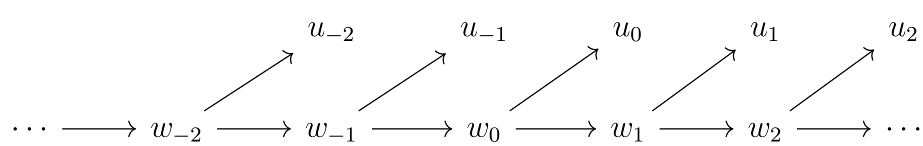

Figure 1. A non-antiwellfounded frame on which Grz is valid.

The frames validating Grz have no proper clusters, that is there are no distinct mutually accessible worlds. Logics with such a property are often referred to as of *fatness* 1. It is easy to misconstrue this, however, and this mistake occurs in the literature [Kra99], that Grz is characterized by reflexive frames of fatness 1. The subtlety lies in the fact that Grz is characterized by reflexive *finite* frames of fatness 1. In the infinite case, however, it is not true. Figure below gives an obvious counterexample—a reflexive frame of fatness 1 where .2 is valid but Grz is not.

$$w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots \longrightarrow w_\omega$$

Figure 2. A non-antiwellfounded frame of fatness 1 on which .2 is valid but Grz is not.

3. Characterization by finite lattices

A *labeling* of a frame F for a world u_0 in a Kripke model N validating S4 is an assignment to each node w in F an assertion Φ_w in the propositional modal language, such that:

(1) $(N, u_0) \models \Phi_{w_0}$.
(2) If $(N, u) \models \Phi_w$, then $(N, u) \models \Diamond \Phi_{w'}$ if and only if $w \leq_F w'$.
(3) For every u in N, there is exactly one Φ_w such that $(N, u) \models \Phi_w$.

These properties are expressible in the language of propositional modal logic by the *Jankov-Fine formula* for F:

$$\Box\Big(\bigvee_{w\in F} p_w \wedge \bigwedge_{w\neq v}(p_w \to \neg p_v) \wedge \bigwedge_{w\leq v}(p_w \to \Diamond p_v) \wedge \bigwedge_{w>v}(p_w \to \neg\Diamond p_v)\Big).$$

Note that the labeling of a frame can be naturally carried out for statements in any language, not necessarily just propositional modal assertions, as long as we have a coherent notion of modal truth for that language. This has been applied extensively in works on set-theoretic, model-theoretic, and category-theoretic potentialism [Woł], [HW20], [HL19], and [HLL15]. The proof of the following lemma can be found in [HLL15], where the frame labeling was coined.

Labeling Lemma. *Suppose that $w \mapsto \Phi_w$ is a labeling of a finite frame F with an initial node w_0 for a world u_0 in a Kripke model N validating* S4*. Then for any Kripke model M based on F there is an assignment of the propositional variables to modal assertions $p \mapsto \psi_p$ such that for any modal assertion $\varphi(p_0, \ldots, p_k)$,*

$$(M, w_0) \models \varphi(p_0, \ldots, p_k) \quad \textit{if and only if} \quad (N, u_0) \models \varphi(\psi_{p_0}, \ldots, \psi_{p_k}).$$

Let me now introduce the reader to the *control statement theory*. Here, one uses various kinds of control statements—buttons, switches, dials, ratchets, or railyards—to establish upper bounds on the class of modal validities. The methodology of this theory was introduced in series of articles [HL08; HLL15; HL19; HW20].

For the purposes of this article, we only need one kind of a control statement. A *button* is an assertion that is necessarily possibly necessary. Buttons b_i, for $i \in I$, are independent at an initial world u_0 of a Kripke model N if none of the buttons is necessary at u_0 and necessarily, any of the buttons can be *pushed* without affecting the other buttons. Formally, for an $A \subseteq I$, we define $\Theta_A = \bigwedge_{i \in A} \Box b_i \wedge \bigwedge_{i \notin A} \neg \Box b_i$. This asserts that the button pattern is specified exactly by A. The buttons are independent at u_0 if

$$(N, u_0) \models \bigwedge_{i \in I} \neg \Box b_i \wedge \bigwedge_{A \subseteq I} \big(\Box \Theta_A \to \bigwedge_{A^+ \supseteq A} \Diamond \Theta_{A^+} \big).$$

The buttons are all unpushed at first, at u_0, and necessarily, any larger button pattern is realizable.

Lemma 1. *Suppose F is a finite lattice with an initial node w_0 and L is a modal logic containing* S4*. If N is a Kripke model and u_0 a world in it that satisfies L and admits at least $|F|$-many independent buttons, then there exists a labeling of F for u_0.*

Proof. We base our argument on [HL08, Lemma 7.3]. Consider the correspondence $w \mapsto b_w$ between the nodes of F and the independent buttons at u_0. For a $S \subseteq F$, define $b_S = \big(\bigwedge_{s \in S} \Box b_s \big) \wedge \big(\bigwedge_{s \notin S} \neg \Box b_s \big)$, which expresses that only the buttons in S are pushed. For any $w \in F$, let

$$\Phi_w = \bigvee \{ \, b_S \mid w = \bigvee S \, \}.$$

We now show that $w \mapsto \Phi_w$ is a labeling of F for u_0. For every $u \geq_F u_0$, there exists an S such that exactly the buttons b_s for $s \in S$ are pushed at u. So $(N, u) \models \Phi_w$ if and only if $w = \sup S$. And so, precisely one Φ_w can hold for each $u \geq_F u_0$. In particular, $(N, u_0) \models \Phi_{w_0}$, where no buttons are pushed yet. The first two conditions of the definition of a labeling are therefore satisfied.

For the forward direction of the first condition, suppose $(N, u) \models \Diamond \Phi_{w'}$. There exists $u' \models \Phi_{w'}$ that is accessible from u. Therefore, there must be a set S' such that $w' = \bigvee \{ \, b_s \mid s \in S' \, \}$. The buttons pushed at u remain so in all accessible worlds, hence $S \subseteq S'$. As a result, $w = \bigvee S \leq \bigvee S' = w'$.

Conversely, suppose that $u \geq_F u_0$ is such that $(N, u) \models \Phi_w$. Let $S = \{ \, s \in F \mid (N, u) \models b_s \, \}$. In other words, S is the set of indices of buttons that are pushed at u. Note that $w = \sup S$. Consider $w' \geq_F w$. Pushing only $b_{w'}$ at u means accessing some u' where precisely the buttons indexed by the set $S \cup \{w'\}$ are pushed. But $\sup S = w \leq_F w'$, and so $w' = \sup(S \cup \{w'\})$. That means $(N, u') \models \Phi_{w'}$, hence $(N, u) \models \Diamond \Phi_{w'}$ □

A *baled tree* is a partially ordered set T that has the greatest element $t \in T$, with the property that when this element is removed, the remaining set $T \setminus t$ is a tree. The idea behind a baled tree is to imagine an upward-growing tree with its topmost part gathered together and tied into a bale.

Lemma 2. *A modal assertion that is not provable in the modal logic* Grz.2 *has to fail in some Kripke model whose frame is a finite baled tree, and hence a finite lattice.*

Proof. We follow the ideas developed in [HL08, Lemma 6.5]. The logic Grz.2 is characterized by finite directed partial orders. Therefore, if Grz.2 $\nvdash \varphi$, then there is a Kripke model N based on a directed partially ordered frame F, together with a world u_0 in it such that $(N, u_0) \not\models \varphi$. We will construct a Kripke model M that is a finite baled tree and bisimilar to N. We use the technique of *partial tree unraveling*, cf. the proof of [HL08, Lemma 6.5].

We say that t is a *path* from u_0 to u in F if and only if it is a maximal subset of the interval $[u_0, u]$ with respect to $\leq_F$. Let $T = \{ t \mid t \text{ is a path from } u_0 \text{ to } u \}$ endowed with a partial order $\leq_T$ where $t_0 \leq_T t_1$ if and only if t_1 is an end-extension of t_0, so that T forms a tree. Let B be the partial tree unraveling of F. Specifically, B is such that B contains the greatest element g of F and all pairs $\langle u, t \rangle$ where t is a path from u_0 to u, with $u \leq g$, and $\langle u, t \rangle \leq_B \langle u', t' \rangle$ if and only if $u \leq_F u'$ and $t \leq_T t'$. Observe that B forms a baled tree, with g being its greatest element.

We now construct the desired Kripke model M based on B. Namely, we copy the values of propositional variables from each world $u \in N$ to all its copies $\langle u, t \rangle$ of itself in F. Each world in M accesses precisely the worlds that are copies of the worlds it accesses in N. Therefore, M and N are bisimilar via the defined correspondence. As a result, every world in M has exactly the same modal truths as its copies in N. Consequently, φ fails in M at a world that is a copy of w_0. Thus, φ fails in a Kripke model with a frame that is a finite baled tree, and consequently, a lattice. □

The following result demonstrates that arbitrarily finitely many independent buttons suffice to bound the propositional modal validities by Grz.2. I improve this result to the context of potentialist systems or Kripke categories in [Woł], answering a longstanding open question concerning buttons and switches, as discussed in [Pas19], cf. [Woł22].

Theorem 3. *Suppose u_0 is an initial world in a Kripke model N that admits arbitrarily finitely many independent buttons. Then, the propositional modal assertions valid at u_0 are contained in the modal logic* Grz.2.

Proof. Parts of the proof are reminiscent to [HL08, Lemma 9]. Suppose Grz.2 $\nvdash \varphi$. By lemma 2, there is a Kripke model M with an initial world w_0 based on a frame that is a finite lattice such that $(M, w_0) \not\models \varphi$. By the labeling lemma and lemma 1, we can assign propositional variables to modal assertions $p \mapsto \psi_p$ so that

$$(M, w_0) \models \varphi(p_0, \ldots, p_n) \quad \text{if and only if} \quad (N, u_0) \models \varphi(\psi_{p_0}, \ldots, \psi_{p_k}).$$

But $(M, w_0) \models \neg\varphi$, so $(N, u_0) \models \neg\varphi(\psi_{p_0}, \ldots, \psi_{p_k})$. And so, φ cannot be valid at u_0. As φ was chosen arbitrarily, any modal assertion not provable in Grz.2 cannot be valid at u_0. □

4. Characterization by finite Boolean algebras

Lemma 4. *For any $n < \omega$, there is a Kripke model N with an initial world u_0 whose frame is a finite Boolean algebra, such that u_0 admits n-many independent buttons.*

Proof. We proceed like in [HL08, Lemma 8]. Take the frame to be the powerset of n ordered by inclusion. This is a finite Boolean algebra, whose nodes are subsets B of n. Now let N be a Kripke model based on that frame where b_i is true just in case $i \in B$. Clearly, every b_i is a button, and—since whatever the button-pattern B any world has, any larger pattern $B' \supseteq B$ is possible—they are all independent at u_0. □

Recall that a modal logic is *sound* with respect to a class of frames if it is valid in each frame from that class. The following is an analogue of [HL08, Lemma 9] but with no switches.

Lemma 5. *A class of frames characterizes the modal logic* Grz.2 *if and only if* Grz.2 *is sound with respect to that class of frames and there are Kripke models based on frames from the class whose initial worlds admit arbitrarily finitely many independent buttons.*

Proof. The forward implication is immediate by lemma 4, showing that Grz.2 is consistent with the existence of arbitrarily finitely many independent buttons. The converse follows from theorem 3. □

Let me conclude the paper with a theorem that summarizes the frame characterizations for the modal logic Grz.2.

Theorem 6. *The following sets of frames characterize the modal logic* Grz.2*.*

(1) Finite directed partial orders.
(2) Finite latices.
(3) Finite baled trees.
(4) Finite Boolean algebras.

Proof. Statement (1) is clear by the original characterization of the logic. Lemma 2 proves that everything that is not provable in Grz.2 fails at a world in a Kripke model based on a finite baled tree (hence a finite lattice and a finite partial order). Lemma 4 shows that the set of finite Boolean algebras have Kripke models with initial worlds admitting arbitrarily finitely many independent buttons. And so, by lemma 5 this class characterizes the modal logic Grz.2. □

References

[Fin74] Kit Fine. "Logics Containing K4. Part I". *The Journal of Symbolic Logic* 39.1 (1974), pp. 31–42. ISSN: 00224812. http://www.jstor.org/stable/2272340 (version 16 February 2023).

[Fin85] Kit Fine. "Logics Containing K4. Part II". *The Journal of Symbolic Logic* 50.3 (1985), pp. 619–651. ISSN: 00224812. http://www.jstor.org/stable/2274318 (version 16 February 2023).

[HL08] Joel David Hamkins and Benedikt Löwe. "The modal logic of forcing". *Trans. AMS* 360.4 (2008), pp. 1793–1817. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-07-04297-3. arXiv:math/0509616[math.LO]. http://wp.me/p5M0LV-3h.

[HL19] Joel David Hamkins and Øystein Linnebo. "The modal logic of set-theoretic potentialism and the potentialist maximality principles". *Review of Symbolic Logic* (October 2019). DOI: 10.1017/S1755020318000242. arXiv:1708.01644[math.LO]. http://wp.me/p5M0LV-1zC.

[HLL15] Joel David Hamkins, George Leibman, and Benedikt Löwe. "Structural connections between a forcing class and its modal logic". *Israel Journal of Mathematics* 207.2 (2015), pp. 617–651. ISSN: 0021-2172. DOI: 10.1007/s11856-015-1185-5. arXiv:1207.5841[math.LO]. http://wp.me/p5M0LV-kf.

[HW20] Joel David Hamkins and Wojciech Aleksander Wołoszyn. "Modal model theory". *Mathematics ArXiv* (2020). Under review. arXiv:2009.09394[math.LO].

[Kra99] Marcus Kracht. *Tools and Techniques in Modal Logic.* Elsevier, 1999.

[Pas19] Robert Passmann. *Are buttons really enough to bound validities by S4.2?* MathOverflow. 2019. https://mathoverflow.net/q/339844.

[Sol76] Robert M. Solovay. "Provability Interpretations of Modal Logic". *Israel Journal of Mathematics* 25 (1976), pp. 287–304.

[Woł] Wojciech Aleksander Wołoszyn. "The modal theory of the category of sets". email circulated preprint.

[Woł22] Wojciech Aleksander Wołoszyn. *Are buttons really enough to bound validities by S4.2?* MathOverflow. 2022. https://mathoverflow.net/q/426422.

(Wojciech Aleksander Wołoszyn) Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, United Kingdom & St Hilda's College, Cowley Place, Oxford, OX4 1DY, United Kingdom

Email address: wojciech.woloszyn@maths.ox.ac.uk

URL: https://woloszyn.org

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